Topological methods in combinatorics: ham sandwich theorem*

There is a sandwich, made of ham, cheese and bread. Two hungry people want to split it. To lose no time, they wish to make a single cut with a knife so that each of the ingredients is split equally between the two halves. It turns out this is possible, and the result is known as the ham-sandwich theorem. It asserts that in arbitrary dimension d it is possible to split d ingredient by a single hyperplane.

A measure μ on a set X is *finite* if $\mu(X) < \infty$. A measure is *Borel* if all open sets are measurable, and consequently all the sets in the σ-algebra generated by the open sets are measurable (the sets in this σ-algebra are called Borel sets). A Borel measure is outer regular if $\mu(S) = \inf_{\substack{S\subset U\\ U \text{ open}}} \mu(U)$.

Theorem 1 (Ham-sandwich theorem)**.** *Suppose $\mu_1, \dots, \mu_d$ are d finite outer regular Borel measure on $\mathbb{R}^d$. Then there is a hyperplane $H = \{x : \langle x, v\rangle = r\}$ such that the two closed halfspace that it bounds, $H_+ = \{x : \langle x, v\rangle \geq r\}$ and $H_- = \{x : \langle x, v\rangle \leq r\}$, satisfy*

$$\mu_i(H_+) \geq \tfrac{1}{2}\mu(\mathbb{R}^d), \text{ and } \mu_i(H_-) \geq \tfrac{1}{2}\mu(\mathbb{R}^d) \quad \text{for all } i = 1, \dots, d.$$

The reason why the theorem does not assert that $\mu_i(H_+) = \mu_i(H_-) = \frac{1}{2}\mu(\mathbb{R}^d)$ is because the measure might be concentrated on a single point, or more generally on a finite set. Of course, if the measure of every hyperplane is zero, then $\mu_i(H_+) = \mu_i(H_-) = \frac{1}{2}\mu(\mathbb{R}^d)$ because $\mu_i(H) = \mu_i(H_+ \cap H_-) = 0$.

A commonly used special case of Theorem 1 is the case where each measure μ_i is a sum of point masses, in which case we obtain the following:

Corollary 2. *If $A_1, \dots, A_d$ are finite sets (or even multisets) in $\mathbb{R}^d$, then there is a hyperplane H such that*

$$|A_i \cap H_+| \geq \tfrac{1}{2}|A_i|, \text{ and } |A_i \cap H_-| \geq \tfrac{1}{2}|A_i| \quad \text{for all } i = 1, \dots, d.$$

Our strategy to prove Theorem 1 is to prove it for the special case of nice measures, and use a limiting argument to deduce the general case. Measures μ and λ are equicontinuous if μ is absolutely continuous with respect to λ and λ is absolutely continuous with respect to μ, in other words $\mu(S) = 0 \iff \lambda(S) = 0$ for every set S.

*These notes are from `http://www.borisbukh.org/TopCombLent12/notes_hamsandwich.pdf`.

Theorem 3. *Suppose $\mu_1, \dots, \mu_d$ are d finite Borel measures on $\mathbb{R}^d$ that are equicontinuous with the Lebesgue measure on $\mathbb{R}^d$. Then there is a hyperplane H as in the preceding theorem, so that*

$$\mu_i(H_+) = \mu_i(H_+) = \tfrac{1}{2}\mu_i(\mathbb{R}^d).$$

Proof of Theorem 3. For $v \in S^{d-1}$ and $r \in \mathbb{R}$ let $H(v,r) \stackrel{\text{def}}{=} \{x : \langle x, v\rangle = r\}$, $H_+(v,r) \stackrel{\text{def}}{=} \{x : \langle x, v\rangle \geq r\}$ and $H_-(v,r) \stackrel{\text{def}}{=} \{x : \langle x, v\rangle \leq r\}$ be the hyperplane parametrised by (v,r) and the two closed halfspaces that it bounds.

As a function of r the function $\mu_d\big(H_-(v,r)\big)$ is strictly increasing because $H_-(r,v) \subset H_-(v,r')$ if $r < r'$ and the difference $H_-(v,r') \setminus H_-(r,v)$ contains an open set, and hence has positive measure. Furthermore, $\mu_d\big(H_-(v,r)\big)$ is a continuous function of r. Indeed, on one hand

$$\mu_d\big(H_-(v,r) \setminus H_-(v, r-\varepsilon)\big) \to \mu_d\big(H(v,r)\big) = 0$$

because μ_d is absolutely continuous with respect to the Lebesgue measure and $H(v,r)$ has zero Lebesgue measure. On the other hand, $\mu_d\big(H_-(v,r+\varepsilon)\setminus H_-(v,r)\big) \to \mu_d(\emptyset) = 0$.

The preceding discussion of $\mu_d\big(H_-(v,r)\big)$ extends to $\mu_d\big(H_+(v,r)\big)$, showing that the latter is a strictly decreasing and continuous function of r. As

$$\lim_{r\to\infty} \mu_d\big(H_-(v,r)\big) = \mu_d(\mathbb{R}^d), \qquad \lim_{r\to-\infty} \mu_d\big(H_-(v,r)\big) = 0,$$
$$\lim_{r\to\infty} \mu_d\big(H_+(v,r)\big) = \mu_d(\mathbb{R}^d), \qquad \lim_{r\to-\infty} \mu_d\big(H_-(v,r)\big) = 0$$

for each v there is a unique $r_{\text{eq}} = r_{\text{eq}}(v)$ such that $\mu_d(H_-(v, r_{\text{eq}})) = \mu_d(H_+(v, r_{\text{eq}}))$. Since $\mu(H_- \cap H_+) = 0$, we furthermore have $\mu_d(H_-(v, r_{\text{eq}})) = \mu_d(H_+(v, r_{\text{eq}})) = \frac{1}{2}\mu_d(\mathbb{R}^d)$. Define $g\colon S^{d-1} \to \mathbb{R}^{d-1}$ by

$$g(v)_i \stackrel{\text{def}}{=} \mu_i\big(H_-(v, r_{\text{eq}})\big) - \mu_i\big(H_+(v, r_{\text{eq}})\big).$$

If g is a continuous function, then by Borsuk–Ulam theorem, g vanishes for some v. For that value of v, the hyperplane $H(v, \text{eq})$ satisfies the conclusion of the theorem because $g(v)_0 = 0$ implies $\mu_i\big(H_-(v, r_{\text{eq}})\big) = \mu_i\big(H_+(v, r_{\text{eq}})\big)$.

So, we need to establish continuity of g. It suffices to establish the continuity of $r_{\text{eq}}(v)$ as a function of v and of $h_-(u,v) = \mu_i\big(H_-(v,r)\big)$ and $h_+(u,v) = \mu_i\big(H_+(v,r)\big)$ as functions of (v,r). Due to the kinship of h_- and h_+, we demonstrate only the continuity of h_-, for the continuity of h_+ is similar.

Continuity of $h_-(v,r)$: Suppose $(v_n, r_n) \to (v,r)$, and we wish to show that $h_-(v_n, r_n) \to (v,r)$. Let χ_n be the characteristic function of $H_-(v_n, r_n)$, and χ the characteristic function of $H_-(v,r)$. Then χ_n converges to χ pointwise everywhere in $\mathbb{R}^d$ except $H(v,r)$. Since $H(v,r)$ is a null set according to μ_d, by the dominated convergence theorem $h_-(v_n, r_n) = \int \chi_n \, d\mu_d \to \int \chi \, d\mu_d = h_-(v,r)$.

Continuity of $r_{\text{eq}}(v)$: Let $\Delta(v,r) = h_-(v,r) - h_+(v,r)$. By the above, Δ is a continuous function, and so $r_{\text{eq}}(v) = \min\{r : \Delta(v,r) = 0\}$ is continuous. □

Proof that Theorem 3 implies Theorem 1. Let $\mu_1, \dots, \mu_d$ be as in Theorem 1. Denote the Lebesgue measure on $\mathbb{R}^d$ by λ. Let $\mu_i^{(n)} = \mu_i * f_n$, where the convolution of a measure and a function is defined by

$$\mu * f(U) \stackrel{\text{def}}{=} \iint_{x+y\in U} f(x)\, d\lambda(x) d\mu(y) \quad \text{for every Borel set } U, \tag{1}$$

and $\{f_n\}_{n=1}^\infty$ is an approximate identity such that f_n is everywhere positive for each n. For instance, we may take

$$f_n = c_n \exp(-n|x|^2)$$

where the constant c_n is chosen so that $\int f_n\, d\lambda = 1$ (or explicitly, $c_n = (\pi/n)^{d/2}$).

On one hand, the Lebesgue measure is absolutely continuous with respect to $\mu_i^{(n)}$ because for each set U the definition (1) is an integral of an everywhere positive function on a set of positive measure. On the other hand, the measure μ_i is absolutely continuous with respect to the Lebesgue measure because if $\lambda(U) = 0$, then if we integrate over x first, the inner integral vanishes, leading to $\mu * f(U) = 0$. By the Theorem 3 for each n there is a hyperplane $H^{(n)}$ so that $\mu_i^{(n)}(H_+^{(n)}) = \mu_i^{(n)}(H_+^{(n)}) = \frac{1}{2}\mu_i^{(n)}(\mathbb{R}^d)$. Our next step is to extract a convergent subsequence of these hyperplanes, but the difficulty is that the space of all hyperplanes in $\mathbb{R}^d$ is not compact. We remedy this by showing that all $H^{(n)}$, for n large, meet a specific ball in $\mathbb{R}^d$.

Let R be large enough so that the ball $B(0,R)$ centred at the origin of radius R satisfies $\mu_1(\mathbb{R}^d \setminus B) < \frac{1}{8}\mu_1(\mathbb{R}^d)$. Let n be so large that $\int_{|x|\geq R} f(x) \leq \frac{1}{8}$. Then

$$\begin{aligned}
\mu_1^{(n)}\big(\mathbb{R}^d \setminus B(0,2R)\big) &= \iint_{|x+y|\geq 2R} f(x)\, d\lambda(x) d\mu_1(y) \\
&\leq \left[\iint_{|y|\geq R} + \iint_{|x|\geq R}\right] f(x)\, d\lambda(x) d\mu_1(y) \\
&= \mu_1(\mathbb{R}^d \setminus B(0,R)) + \mu_1(\mathbb{R}^d) \int_{|x|\geq R} f(x)\, dx \\
&\leq \tfrac{1}{4}\mu_1(\mathbb{R}^d).
\end{aligned}$$

Thus, for all large enough n the hyperplane $H^{(n)}$ meets the ball $B(0,2R)$. Indeed, if $H^{(n)}$ did not meet $B(0,2R)$, one of the two halfspace, $H_-^{(n)}$ or $H_+^{(n)}$, would be contained in $\mathbb{R}^d \setminus B(0,2R)$, and would have measure less than $\frac{1}{4}\mu_1(\mathbb{R}^d)$.

The space of all the hyperplanes meeting $B(0,2R)$ is just $\{H(v,r) : v \in S^{d-1},\ r \leq 2R\}$, and so is compact. Thus, there is a convergent subsequence of $H^{(1)}, H^{(2)}, \dots,$ that converges to some hyperplane H. Without loss of generality, but with a gain in notation, we assume that this subsequence is $H^{(1)}, H^{(2)}, \dots$ itself.

By a change in coordinate system, we may assume that $H = \{x_d = 0\}$. Let

$\varepsilon > 0$ be arbitrary. Suppose n is so large that $\int_{|x|\geq\varepsilon} f(x) \leq \varepsilon/\mu_i(\mathbb{R}^d)$. Then

$$\begin{aligned}
\mu_i^{(n)}(\{x : x_d \geq 0\}) &= \iint_{x_d+y_d\geq 0} f(x)\, d\lambda(x) d\mu_i(y) \\
&\leq \left[\iint_{y_d\geq -\varepsilon} + \iint_{x_d\geq\varepsilon}\right] f(x)\, d\lambda(x) d\mu_i(y) \\
&= \mu_i(\{y : y_d \geq -\varepsilon\}) + \mu_i(\mathbb{R}^d) \int_{x_d\geq\varepsilon} f(x)\, dx \\
&\leq \mu_i(\{y : y_d \geq -\varepsilon\}) + \varepsilon.
\end{aligned}$$

Since $\mu_i^{(n)}(\mathbb{R}^d) = \mu_i(\mathbb{R}^d)$ (exercise!), it thus follows that

$$\mu_i(\{y : y_d \geq -\varepsilon\}) \geq \tfrac{1}{2}\mu_i(\mathbb{R}^d) - \varepsilon.$$

Since ε is arbitrary and μ_i is outer regular, we conclude that $\mu_i(\{y : y_d \geq 0\}) \geq \frac{1}{2}\mu_i(\mathbb{R}^d)$. Since the same is true for $\{y : y_d \leq 0\}$, the theorem holds. □