Extremal Combinatorics: homework #2*
Due 21 February 2026, at 10pm

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Suppose that H is a graph with $\chi(H) = 3$. Prove that there is a constant $\varepsilon = \varepsilon(H) > 0$ such that $\operatorname{ex}(n, H) \le \frac{1}{4}n^2 + O(n^{2-\varepsilon})$.

2. Find a graph H with $\chi(H) = 3$ and $\operatorname{ex}(n, H) \ge \frac{1}{4}n^2 + n^{1.99}$.

3. Show that there exists a 3-regular n-vertex graph of girth $\Omega(\log n)$ for all large even n.

4. Call a subset X of abelian group Γ an *(s,t)-set* if it is of the form $X = A + B$ for some $A, B \subset \Gamma$ of sizes $|A| = s$ and $|B| = t$. Furthermore, call it *proper (s,t)-set* if $|X| = st$.

   (a) Show that there is function $f_s(t)$ satisfying $\lim_{t\to\infty} f_s(t) = \infty$ such that every (s,t)-set contains a proper $(s, f(t))$-set.

   (b) Show that there is a function $t(s)$ such that, whenever $p \ge p_0(s)$ is a sufficiently large prime, there is a set $S \subset \mathbb{F}_p^s$ of size $|S| \ge \frac{1}{2}p^{s-1}$ that contains no $(s, t(s))$-set.

   (c) Deduce that, for all sufficiently large $N \ge N_0(s)$, the set $[N]$ contains a $(s, t(s))$-good subset of size $\Omega_s(N^{1-1/s})$.

More problems will be added February 13.

*This homework is from `http://www.borisbukh.org/ExtremalCombinatorics26/hw2.pdf`.