Set families with a forbidden subposet

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August 2008


## Introduction

$$
[n]=\{1, \ldots, n\} \quad 2^{[n]}=\{F: F \subset[n]\}
$$

## Definition

Set family is a collection of subsets of [n]. In symbols, $\mathcal{F} \subset 2^{[n]}$.

## Theorem (Sperner'28)

Suppose $\mathcal{F} \subset 2^{[n]}$ is a set family such that for no distinct $F_{1}, F_{2} \subset \mathcal{F}$ the inclusion $F_{1} \subset F_{2}$ holds. Then

$$
|\mathcal{F}| \leq\binom{ n}{\lfloor n / 2\rfloor} .
$$



## Reformulation

## Definition

$P_{1}$ is a subposet of $P_{2}$ (written $P_{1} \subset P_{2}$ ) if there is an injective $f: P_{1} \rightarrow P_{2}$ such that

$$
x<_{p_{1}} y \Longrightarrow f(x)<_{p_{2}} f(y)
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Examples:

and


Way of thinking
A set family is a poset under inclusion. For $F_{1}, F_{2} \in \mathcal{F}$ set $F_{1} \leq_{\mathcal{F}} F_{2}$ if $F_{1} \subset F_{2}$.

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Theorem (Sperner'28)
$: \not \subset \mathcal{F} \Longrightarrow|\mathcal{F}| \leq\binom{ n}{\lfloor n / 2\rfloor}$

## Reformulation and the known results

## Notation

Largest family in [n] not containing poset $P$ has size

$$
\operatorname{ex}(P, n)=\max _{\substack{P \not \subset \mathcal{F} \\ \mathcal{F} \subset 2^{[n]}}}|\mathcal{F}|
$$

Sperner'28

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\begin{aligned}
& \operatorname{ex}(!, n)=\binom{n}{\lfloor n / 2\rfloor} \\
& \operatorname{ex}\binom{!}{\lfloor }=\binom{n}{\lfloor n / 2\rfloor}+\binom{n}{\lfloor n / 2\rfloor+1} \\
& \operatorname{ex}(\because, n)=\binom{n}{n / 2}(1+O(1 / n)) \\
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Erdős'45
Katona-Tarján'83
Thanh'98
De Bonis-Katona-Swanepoel'05 Griggs-Katona’08

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n \\
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## Conjecture

For a fixed poset $P$

$$
\operatorname{ex}(P, n)=I(P)\binom{n}{n / 2}(1+O(1 / n))
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where $I(P)$ is the largest number of "middle" levels whose union contains no $P$.

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Qualitatively explains all the previously known result including : poset because $\therefore \subset \chi$
For $h(P)=2$ independently proved by Griggs and Lu. They also proved the conjecture for a large class of posets with $h(P)=2$, whose Hasse diagram is not a tree.

## First idea for $h(P)=2$

Have: $|\mathcal{F}| \geq(1+\varepsilon)\binom{n}{n / 2}$, poset $P=:$
Want: an embedding of $P$ into poset $(\mathcal{F}, \subset)$

## Idea

Treat poset $(\mathcal{F}, \subset)$ as a graph, and embed tree into it.

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## Trouble with $h(P) \geq 3$

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For $h(P) \geq 3$ instead of graphs have to use $h(P)$-uniform hypergraphs, and there is no good analogue of minimum degree.

## Problem

How to embed a tree into a graph of large average degree without using minimum degree?

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For every tree $T$ there is a $d=d(T)$ such that every graph $G$ of average degree $\geq d$ contains $T$.

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## Proof.

Induction on $|T|$. If $|T|=1$, trivial. Let $v$ be a leaf. Else let $T^{\prime}=T \backslash\{v\}$, and $d(T)=2 d\left(T^{\prime}\right)+4|T|$

- Let $V^{\prime}=\{x \in V(G): \operatorname{deg}(x) \geq d(T) / 4\}$.

Define $G^{\prime}=\left.G\right|_{V^{\prime}}$

- Average degree of $G^{\prime}$ is at least $d(T) / 2$.
- Find an embedding of $T^{\prime}$ into $G^{\prime}$.

- Since $\operatorname{deg}(u) \geq|T|$ in $G$, extend the embedding.

