Set families with a forbidden subposet

Boris Bukh

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\[ [n] = \{1, \ldots, n\} \quad 2^{[n]} = \{F : F \subset [n]\} \]

**Definition**

*Set family* is a collection of subsets of \([n]\). In symbols, \(\mathcal{F} \subset 2^{[n]}\).

**Theorem (Sperner’28)**

Suppose \(\mathcal{F} \subset 2^{[n]}\) is a set family such that for no distinct \(F_1, F_2 \subset \mathcal{F}\) the inclusion \(F_1 \subset F_2\) holds. Then

\[ |\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}. \]

*Equality is attained for \(\mathcal{F} = \binom{[n]}{\lfloor n/2 \rfloor} = \{F \subset [n] : |F| = \lfloor n/2 \rfloor\}.*
Reformulation

Definition

$P_1$ is a subposet of $P_2$ (written $P_1 \subset P_2$) if there is an injective $f : P_1 \rightarrow P_2$ such that

\[ x \prec_{P_1} y \implies f(x) \prec_{P_2} f(y) \]

Examples:

\[
\begin{align*}
\begin{array}{c}
  a \\
  b
\end{array} & \subset \\
\begin{array}{c}
  a \\
  b
\end{array} & \begin{array}{c}
  a \\
  b
\end{array} \\
\begin{array}{c}
  b \\
  c
\end{array} & \begin{array}{c}
  b \\
  c
\end{array}
\end{align*}
\]

and

Way of thinking

A set family is a poset under inclusion. For $F_1, F_2 \in \mathcal{F}$ set $F_1 \leq_{\mathcal{F}} F_2$ if $F_1 \subset F_2$. 
Definition

$P_1$ is a subposet of $P_2$ (written $P_1 \subset P_2$) if there is an injective $f : P_1 \rightarrow P_2$ such that

$$x <_{P_1} y \implies f(x) <_{P_2} f(y)$$

Examples:

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Examples:

\[ a \subset b \]
\[ a \subset b \]
\[ a \subset b \]

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Examples:

$$\begin{array}{c}
\text{and}
\end{array}$$

Way of thinking

A set family is a poset under inclusion. For $F_1, F_2 \in \mathcal{F}$ set $F_1 \leq_{\mathcal{F}} F_2$ if $F_1 \subset F_2$. 

Diagram:
Definition

$P_1$ is a subposet of $P_2$ (written $P_1 \subset P_2$) if there is an injective $f : P_1 \to P_2$ such that

$$x <_{P_1} y \implies f(x) <_{P_2} f(y)$$

Examples:

$$\begin{align*}
\begin{tikzpicture}[scale=0.5]
    \node [shape=circle,draw=black] (a) at (0,2) {a};
    \node [shape=circle,draw=black] (b) at (0,0) {b};
    \end{tikzpicture}
\end{align*} \subset
\begin{align*}
\begin{tikzpicture}[scale=0.5]
    \node [shape=circle,draw=black] (a) at (0,2) {a};
    \node [shape=circle,draw=black] (b) at (1,1) {b};
    \node [shape=circle,draw=black] (c) at (2,0) {c};
    \end{tikzpicture}
\end{align*}
\quad
\text{and}
\quad
\begin{align*}
\begin{tikzpicture}[scale=0.5]
    \node [shape=circle,draw=black] (a) at (0,2) {a};
    \node [shape=circle,draw=black] (b) at (1,1) {b};
    \node [shape=circle,draw=black] (c) at (2,0) {c};
    \end{tikzpicture}
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\begin{tikzpicture}[scale=0.5]
    \node [shape=circle,draw=black] (a) at (0,2) {a};
    \node [shape=circle,draw=black] (b) at (1,1) {b};
    \end{tikzpicture}
\end{align*}$$

Way of thinking

A set family is a poset under inclusion. For $F_1, F_2 \in \mathcal{F}$ set

$F_1 \leq_\mathcal{F} F_2$ if $F_1 \subset F_2$. 
Definition

$P_1$ is a subposet of $P_2$ (written $P_1 \subset P_2$) if there is an injective $f : P_1 \rightarrow P_2$ such that

$$x \leq_{P_1} y \iff f(x) \leq_{P_2} f(y)$$

Examples: $\subset$ and $\subset$

Way of thinking

A set family is a poset under inclusion. For $F_1, F_2 \in \mathcal{F}$ set $F_1 \leq_{\mathcal{F}} F_2$ if $F_1 \subset F_2$. 
Reformulation

Definition

$P_1$ is a subposet of $P_2$ (written $P_1 \subset P_2$) if there is an injective $f : P_1 \rightarrow P_2$ such that

$\forall x, y \in P_1$ such that $x \leq P_1 y$ implies $f(x) \leq P_2 f(y)$.

Examples:

![Diagram showing subposets](image)

Way of thinking

A set family is a poset under inclusion. For $F_1, F_2 \in \mathcal{F}$ set $F_1 \leq_\mathcal{F} F_2$ if $F_1 \subset F_2$. 
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$P_1$ is a subposet of $P_2$ (written $P_1 \subset P_2$) if there is an injective $f : P_1 \to P_2$ such that

$\forall x, y \in P_1 : x <_{P_1} y \implies f(x) <_{P_2} f(y)$

Examples:

Way of thinking

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Way of thinking
Reformulation

Examples: \( \begin{array}{c}
\bullet a \\
\bullet b \\
\end{array} \quad \subset \quad \begin{array}{c}
\bullet a \\
\bullet b \\
\bullet c \\
\end{array} \quad \text{and} \quad \begin{array}{c}
\bullet a \\
\bullet b \\
\bullet c \\
\end{array} \quad \subset \quad \begin{array}{c}
\bullet a \\
\bullet b \\
\bullet c \\
\end{array} \end{array} \)

Way of thinking

A set family is a poset under inclusion. For \( F_1, F_2 \in \mathcal{F} \) set \( F_1 \leq_{\mathcal{F}} F_2 \) if \( F_1 \subset F_2 \).
**Reformulation**

**Definition**

$P_1$ is a subposet of $P_2$ (written $P_1 \subset P_2$) if there is an injective $f: P_1 \rightarrow P_2$ such that $x \prec P_1 y \Rightarrow f(x) \prec P_2 f(y)$.

**Examples:**

$\subset$ and $\subset$

**Way of thinking**

A set family is a poset under inclusion. For $F_1, F_2 \in \mathcal{F}$ set $F_1 \leq \mathcal{F} F_2$ if $F_1 \subset F_2$. 

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Examples:

$$
\begin{align*}
\text{ and } \\
\begin{array}{c}
\bullet \quad \subset \quad \bullet \\
b & a \\
\end{array} & \quad \text{ and } & \quad \begin{array}{c}
\bullet \quad \subset \quad \bullet \\
b & a \\
\end{array} & \quad \begin{array}{c}
\bullet \quad \subset \quad \bullet \\
b & c \\
\end{array}
\end{align*}
$$

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A set family is a poset under inclusion. For $F_1, F_2 \in \mathcal{F}$ set $F_1 \leq_{\mathcal{F}} F_2$ if $F_1 \subset F_2$. 
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**Examples:**

\[
\begin{array}{ccc}
\text{Examples:} & \subset & \text{and} \\
\begin{array}{ccc}
 a \\
 b
\end{array} & \subset & \begin{array}{ccc}
 a \\
 b \\
 b
\end{array} \\
 a & \subset & \begin{array}{ccc}
 a \\
 b \\
 c \\
 b \\
 c
\end{array}
\end{array}
\]

**Way of thinking**

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A set family is a poset under inclusion. For $F_1, F_2 \in \mathcal{F}$ set $F_1 \leq_{\mathcal{F}} F_2$ if $F_1 \subset F_2$.

Theorem (Sperner’28)

$\emptyset \not\in \mathcal{F} \implies |\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$
Reformulation and the known results

**Notation**

Largest family in $[n]$ not containing poset $P$ has size

$$\text{ex}(P, n) = \max_{P \not\subset F} |F|$$

$F \subseteq 2^{[n]}$

<table>
<thead>
<tr>
<th>Year</th>
<th>Notation</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sperner’28</td>
<td>$\text{ex}(\emptyset, n)$</td>
<td>$(\binom{n}{\lfloor n/2 \rfloor})$</td>
</tr>
<tr>
<td>Erdős’45</td>
<td>$\text{ex}(\emptyset, n)$</td>
<td>$(\binom{n}{\lfloor n/2 \rfloor}) + (\binom{n}{\lfloor n/2 \rfloor} + 1)$</td>
</tr>
<tr>
<td>Katona–Tarján’83</td>
<td>$\text{ex}(\cup, n)$</td>
<td>$(\binom{n}{n/2})(1 + O(1/n))$</td>
</tr>
<tr>
<td>Thanh’98</td>
<td>$\text{ex}(\Delta, n)$</td>
<td>$2(\binom{n}{n/2})(1 + O(1/n))$</td>
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<tr>
<td>De Bonis–Katona–Swanepoel’05</td>
<td>$\text{ex}(\nabla, n)$</td>
<td>$(\binom{n}{\lfloor n/2 \rfloor}) + (\binom{n}{\lfloor n/2 \rfloor} + 1)$</td>
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<tr>
<td>Griggs–Katona’08</td>
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Reformulation and the known results

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\text{ex}(P, n) = \max_{P \not\subseteq \mathcal{F}} |\mathcal{F}| \\
\mathcal{F} \subseteq 2^{[n]} \]
| Sperner’28                   | \[
\text{ex}(\mathbb{I}, n) = \binom{n}{\lfloor n/2 \rfloor} \]
| Erdős’45                     | \[
\text{ex}(\mathbb{I}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} \]
| Katona–Tarján’83             | \[
\text{ex}(\mathcal{V}, n) = \binom{n}{n/2} (1 + O(1/n)) \]
| Thanh’98                     | \[
\text{ex}(\mathcal{A}, n) = 2 \binom{n}{n/2} (1 + O(1/n)) \]
| De Bonis–Katona–Swanepoel’05 | \[
\text{ex}(\mathcal{K}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} \]
| Griggs–Katona’08             | \[
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<td>$ex(P, n) = \max_{\mathcal{F} \subseteq 2^{[n]}, P \not\subset \mathcal{F}}</td>
<td>\mathcal{F}</td>
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<td>Sperner’28</td>
<td>$ex(\emptyset, n) = \binom{n}{\lfloor n/2 \rfloor}$</td>
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Reformulation and the known results

Notation

Largest family in \([n]\) not containing poset \(P\) has size

\[\text{ex}(P, n) = \max_{P \not\subseteq \mathcal{F}} |\mathcal{F}|\]

<table>
<thead>
<tr>
<th>Sperner’28</th>
<th>(\text{ex}([n/2], n) = \binom{n}{\lfloor n/2 \rfloor})</th>
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<td>Katona–Tarján’83</td>
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<tr>
<td>Thanh’98</td>
<td>(\text{ex}(\exists, n) = 2^{\binom{n}{\lfloor n/2 \rfloor}} (1 + O(1/n)))</td>
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Reformulation and the known results

### Notation

Largest family in \([n] \) not containing poset \(P\) has size

\[
\text{ex}(P, n) = \max_{P \notin \mathcal{F}} |\mathcal{F}|
\]

- **Sperner’28**
  \[
  \text{ex}(\emptyset, n) = \binom{n}{\lfloor n/2 \rfloor}
  \]

- **Erdős’45**
  \[
  \text{ex}(\uparrow, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}
  \]

- **Katona–Tarján’83**
  \[
  \text{ex}(\vee, n) = \binom{n}{n/2} \left(1 + O\left(\frac{1}{n}\right)\right)
  \]

- **Thanh’98**
  \[
  \text{ex}(\downarrow, n) = 2\binom{n}{n/2} \left(1 + O\left(\frac{1}{n}\right)\right)
  \]

- **De Bonis–Katona–Swanepoel’05**
  \[
  \text{ex}(\cup, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}
  \]

- **Griggs–Katona’08**
  \[
  \text{ex}(\cap, n) = \binom{n}{n/2} \left(1 + O\left(\frac{1}{n}\right)\right)
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Reformulation and the known results

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<tr>
<th>Notation</th>
<th>Expression</th>
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<tbody>
<tr>
<td>Largest family in ([n]) not containing poset (P) has size</td>
<td>(\text{ex}(P, n) = \max_{\mathcal{F} \subseteq 2^n}</td>
</tr>
<tr>
<td>Sperner’28</td>
<td>(\text{ex}(\mathcal{I}, n) = \binom{n}{\lfloor n/2 \rfloor})</td>
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<tr>
<td>Erdős’45</td>
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<tr>
<td>Thanh’98</td>
<td>(\text{ex}(\mathcal{L}, n) = 2\binom{n}{\lfloor n/2 \rfloor}(1 + O(1/n)))</td>
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\(\mathcal{I}\): Intervals, \(\mathcal{V}\): Venn diagrams, \(\mathcal{L}\): L-shapes, \(\mathcal{N}\): Non-overlapping.
Reformulation and the known results

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<tr>
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<td>$\max_{P \not\subset \mathcal{F}}</td>
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<td>Katona–Tarján’83</td>
<td>$\binom{n}{n/2} (1 + O(1/n))$</td>
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<td>Thanh’98</td>
<td>$2 \binom{n}{n/2} (1 + O(1/n))$</td>
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<tr>
<td>De Bonis–Katona–Swanepoel’05</td>
<td>$\binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor+1}$</td>
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<td>Griggs–Katona’08</td>
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An explanation for the known results

<table>
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<tr>
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## Notation

Largest family in $[n]$ not containing poset $P$ has size $\text{ex}(P, n) = \max_{F \subseteq 2^{[n]} \setminus P} |F|$

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<td>Katona–Tarján’83</td>
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<td>$\text{ex}(\mathcal{N}, n) = \binom{n}{n/2} \left(1 + O\left(\frac{1}{n}\right)\right)$</td>
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An explanation for the known results

<table>
<thead>
<tr>
<th>Year</th>
<th>Author(s)</th>
<th>Exponential Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>1928</td>
<td>Sperner</td>
<td>(\text{ex}(\mathcal{I}, n) = \binom{n}{\lfloor n/2 \rfloor})</td>
</tr>
<tr>
<td>1945</td>
<td>Erdős</td>
<td>(\text{ex}(\mathcal{I}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1})</td>
</tr>
<tr>
<td>1983</td>
<td>Katona–Tarján</td>
<td>(\text{ex}(\mathcal{V}, n) = \binom{n}{n/2}(1 + O(1/n)))</td>
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<tr>
<td>1998</td>
<td>Thanh</td>
<td>(\text{ex}(\mathcal{M}, n) = 2\binom{n}{n/2}(1 + O(1/n)))</td>
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<tr>
<td>2005</td>
<td>De Bonis–Katona–Swanepoel</td>
<td>(\text{ex}(\mathcal{M}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1})</td>
</tr>
<tr>
<td>2008</td>
<td>Griggs–Katona</td>
<td>(\text{ex}(\mathcal{N}, n) = \binom{n}{n/2}(1 + O(1/n)))</td>
</tr>
</tbody>
</table>
An explanation for the known results

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<tr>
<td>Sperner’28</td>
<td>( \text{ex}(\mathbb{I}, n) = \binom{n}{\lfloor n/2 \rfloor} )</td>
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<tr>
<td>Erdős’45</td>
<td>( \text{ex}(\mathbb{I}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} )</td>
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<tr>
<td>Katona–Tarján’83</td>
<td>( \text{ex}(\mathbb{V}, n) = \binom{n}{n/2} \left( 1 + O(1/n) \right) )</td>
</tr>
<tr>
<td>Thanh’98</td>
<td>( \text{ex}(\mathbb{D}, n) = 2 \binom{n}{n/2} \left( 1 + O(1/n) \right) )</td>
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<tr>
<td>De Bonis–Katona–Swanepoel’05</td>
<td>( \text{ex}(\mathbb{D}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} )</td>
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<td>Griggs–Katona’08</td>
<td>( \text{ex}(\mathbb{N}, n) = \binom{n}{n/2} \left( 1 + O(1/n) \right) )</td>
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### An explanation for the known results

<table>
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<tr>
<th>Author</th>
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<tr>
<td>Sperner ’28</td>
<td>$\text{ex}(\mathbb{I}, n) = \left(\begin{array}{c}n \vspace{1pt} \ \lfloor n/2 \rfloor \end{array}\right)$</td>
</tr>
<tr>
<td>Erdős ’45</td>
<td>$\text{ex}(\mathbb{I}, n) = \left(\begin{array}{c}n \vspace{1pt} \ \lfloor n/2 \rfloor \end{array}\right) + \left(\begin{array}{c}n \vspace{1pt} \ \lfloor n/2 \rfloor + 1 \end{array}\right)$</td>
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<td>Katona–Tarján ’83</td>
<td>$\text{ex}(\mathbb{Y}, n) = \left(\begin{array}{c}n \vspace{1pt} \ n/2 \end{array}\right) \left(1 + O\left(1/n\right)\right)$</td>
</tr>
<tr>
<td>Thanh ’98</td>
<td>$\text{ex}(\mathbb{M}, n) = 2\left(\begin{array}{c}n \vspace{1pt} \ n/2 \end{array}\right) \left(1 + O\left(1/n\right)\right)$</td>
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<td>$\text{ex}(\mathbb{M}, n) = \left(\begin{array}{c}n \vspace{1pt} \ \lfloor n/2 \rfloor \end{array}\right) + \left(\begin{array}{c}n \vspace{1pt} \ \lfloor n/2 \rfloor + 1 \end{array}\right)$</td>
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An explanation for the known results

Sperner’28
\[ \text{ex}(\mathcal{I}, n) = \binom{n}{\lfloor n/2 \rfloor} \]

Erdős’45
\[ \text{ex}(\mathcal{I}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} \]

Katona–Tarján’83
\[ \text{ex}(\mathcal{V}, n) = \binom{n}{n/2}(1 + O(1/n)) \]

Thanh’98
\[ \text{ex}(\mathcal{\mathcal{L}}, n) = 2\binom{n}{n/2}(1 + O(1/n)) \]

De Bonis–Katona–Swanepoel’05
\[ \text{ex}(\mathcal{\mathcal{L}}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} \]

Griggs–Katona’08
\[ \text{ex}(\mathcal{\mathcal{N}}, n) = \binom{n}{n/2}(1 + O(1/n)) \]
An explanation for the known results

Sperner'28  \[ \text{ex}(\mathcal{I}, n) = \binom{n}{\lfloor n/2 \rfloor} \]
Erdős'45  \[ \text{ex}(\mathcal{I}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} \]
Katona–Tarján’83  \[ \text{ex}(\mathcal{V}, n) = \binom{n}{n/2}(1 + O(1/n)) \]
Thanh’98  \[ \text{ex}(\mathcal{H}, n) = 2\binom{n}{n/2}(1 + O(1/n)) \]
De Bonis–Katona–Swanepoel’05  \[ \text{ex}(\mathcal{N}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} \]
Griggs–Katona’08  \[ \text{ex}(\mathcal{N}, n) = \binom{n}{n/2}(1 + O(1/n)) \]

Conjecture

For a fixed poset \( P \)

\[ \text{ex}(P, n) = \ell(P)\binom{n}{n/2}(1 + O(1/n)) \]

where \( \ell(P) \) is the largest number of “middle” levels whose union contains no \( P \).
An explanation for the known results

Sperner’28
\[ \text{ex}(\mathcal{I}, n) = \binom{n}{\lfloor n/2 \rfloor} \]
Erdős’45
\[ \text{ex}(\mathcal{I}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} \]
Katona–Tarján’83
\[ \text{ex}(\mathcal{V}, n) = \binom{n}{n/2} \left(1 + O\left(\frac{1}{n}\right)\right) \]
Thanh’98
\[ \text{ex}(\mathcal{M}, n) = 2\binom{n}{n/2} \left(1 + O\left(\frac{1}{n}\right)\right) \]
De Bonis–Katona–Swanepoel’05
\[ \text{ex}(\mathcal{N}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} \]
Griggs–Katona’08
\[ \text{ex}(\mathcal{N}, n) = \binom{n}{n/2} \left(1 + O\left(\frac{1}{n}\right)\right) \]

Conjecture

For a fixed poset \( P \)
\[ \text{ex}(P, n) = l(P) \binom{n}{n/2} \left(1 + O\left(\frac{1}{n}\right)\right) \]
where \( l(P) \) is the largest number of “middle” levels whose union contains no \( P \).
An explanation for the known results

Sperner’28

\[ \operatorname{ex}(\emptyset, n) = \binom{n}{\lfloor n/2 \rfloor} \]

Erdős’45

\[ \operatorname{ex}(\uparrow, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} \]

Katona–Tarján’83

\[ \operatorname{ex}(\updownarrow, n) = \binom{n}{\lfloor n/2 \rfloor}(1 + O(1/n)) \]

Thanh’98

\[ \operatorname{ex}(\nabla, n) = 2\binom{n}{n/2}(1 + O(1/n)) \]

De Bonis–Katona–Swanepoel’05

\[ \operatorname{ex}(\bigstar, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} \]

Griggs–Katona’08

\[ \operatorname{ex}(\kappa, n) = \binom{n}{\lfloor n/2 \rfloor}(1 + O(1/n)) \]

Conjecture

For a fixed poset \( P \)

\[ \operatorname{ex}(P, n) = l(P)\binom{n}{n/2}(1 + O(1/n)) \]

where \( l(P) \) is the largest number of “middle” levels whose union contains no \( P \).
An explanation for the known results

Sperner’28
ex(\{, n) = \binom{n}{\lfloor n/2 \rfloor}

Erdős’45
ex(\mathcal{I}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor+1}

Katona–Tarján’83
ex(\mathcal{V}, n) = \binom{n}{n/2} \left(1 + O(1/n)\right)

Thanh’98
ex(\mathcal{H}, n) = 2\binom{n}{n/2} \left(1 + O(1/n)\right)

De Bonis–Katona–Swanepoel’05
ex(\mathcal{X}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor+1}

Conjecture

For a fixed poset P

ex(P, n) = l(P) \binom{n}{n/2} \left(1 + O(1/n)\right)

where l(P) is the largest number of “middle” levels whose union contains no P.
An explanation for the known results

Sperner’28  \[ \text{ex}(\mathcal{I}, n) = \binom{n}{\lfloor n/2 \rfloor} \]
Erdős’45  \[ \text{ex}(\mathcal{I}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} \]
Katona–Tarján’83  \[ \text{ex}(\mathcal{V}, n) = \binom{n}{n/2} (1 + O(1/n)) \]
Thanh’98  \[ \text{ex}(\mathcal{\cup}, n) = 2 \binom{n}{n/2} (1 + O(1/n)) \]
De Bonis–Katona–Swanepoel’05  \[ \text{ex}(\mathcal{\cup}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} \]

Conjecture

For a fixed poset \( P \)

\[ \text{ex}(P, n) = l(P) \left( \binom{n}{n/2} \right) (1 + O(1/n)) \]

where \( l(P) \) is the largest number of “middle” levels whose union contains no \( P \).
An explanation for the known results

Sperner’28
ex(\mathcal{I}, n) = \binom{n}{\left\lfloor n/2 \right\rfloor}

Erdős’45
ex(\mathcal{I}, n) = \binom{n}{\left\lfloor n/2 \right\rfloor} + \binom{n}{\left\lfloor n/2 \right\rfloor + 1}

Katona–Tarján’83
ex(\mathcal{V}, n) = \binom{n}{n/2}(1 + O(1/n))

ex(\mathbb{L}, n) = 2\binom{n}{n/2}(1 + O(1/n))

Conjecture

For a fixed poset P

\text{ex}(P, n) = l(P)\binom{n}{n/2}(1 + O(1/n))

where \text{l}(P) is the largest number of “middle” levels whose union contains no P.
An explanation for the known results

Sperner’28

\[ \text{ex}(1, n) = \binom{n}{\lfloor n/2 \rfloor} \]

Erdős’45

\[ \text{ex}(1, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} \]

Katona–Tarján’83

\[ \text{ex}(\mathcal{V}, n) = \binom{n}{n/2} (1 + O(1/n)) \]

\[ \text{ex}(\mathcal{I}, n) = 2 \binom{n}{n/2} (1 + O(1/n)) \]

Thanh’98

\[ \text{ex}(\mathcal{I}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} \]

Conjecture

For a fixed poset \( P \)

\[ \text{ex}(P, n) = l(P) \left( \binom{n}{n/2} (1 + O(1/n)) \right) \]

where \( l(P) \) is the largest number of “middle” levels whose union contains no \( P \).
An explanation for the known results

Sperner’28
ex(\mathcal{I}, n) = \binom{n}{\left\lfloor n/2 \right\rfloor}

Erdős’45
ex(\mathcal{I}, n) = \binom{n}{\left\lfloor n/2 \right\rfloor} + \binom{n}{\left\lfloor n/2 \right\rfloor + 1}

Katona–Tarján’83
ex(\mathcal{Y}, n) = \binom{n}{n/2}(1 + O(1/n))

De Bonis–Katona–Swanepoel’05
ex(P, n) = l(P)\binom{n}{n/2}(1 + O(1/n))

Grimmett–Griggs–Katona’08
ex(P, n) = l(P)\binom{n}{n/2}(1 + O(1/n))

Conjecture

For a fixed poset \( P \)

ex(P, n) = l(P)\binom{n}{n/2}(1 + O(1/n))

where \( l(P) \) is the largest number of “middle” levels whose union contains no \( P \).
An explanation for the known results

Sperner’28
ex(\emptyset, n) = \binom{n}{\lfloor n/2 \rfloor}

Erdős’45
ex(\bullet, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}

Katona–Tarián’83
ex(\vee, n) = \binom{n}{n/2}(1 + O(1/n))

De Bonis–Katona–Swanepoel’05
ex(P, n) = l(P)\binom{n}{n/2}(1 + O(1/n))

Griggs–Katona’08
ex(P, n) = l(P)\binom{n}{n/2}(1 + O(1/n))

Conjecture
For a fixed poset P

ex(P, n) = l(P)\binom{n}{n/2}(1 + O(1/n))

where l(P) is the largest number of “middle” levels whose union contains no P.
Conjecture

For a fixed poset $P$

$$\text{ex}(P, n) = l(P) \binom{n}{n/2} (1 + O(1/n))$$

where $l(P)$ is the largest number of “middle” levels whose union contains no $P$. 

An explanation for the known results

**Conjecture**

For a fixed poset $P$

$$\text{ex}(P, n) = l(P) \binom{n}{n/2} (1 + O(1/n))$$

where $l(P)$ is the largest number of “middle” levels whose union contains no $P$. 
An explanation for the known results

<table>
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<th>Description</th>
<th>Expression</th>
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<tr>
<td>Sperner'28 $\text{ex}(n)$</td>
<td>$\left\lfloor \frac{n}{2} \right\rfloor$</td>
</tr>
<tr>
<td>Erdős'45 $\text{ex}(n)$</td>
<td>$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 1$</td>
</tr>
<tr>
<td>Katona–Tarján'83 $\text{ex}(n)$</td>
<td>$n \left( 1 + O\left( \frac{1}{n} \right) \right)$</td>
</tr>
<tr>
<td>Thanh'98 $\text{ex}(n)$</td>
<td>$2n \left( 1 + O\left( \frac{1}{n} \right) \right)$</td>
</tr>
<tr>
<td>De Bonis–Katona–Swanepoel'05 $\text{ex}(n)$</td>
<td>$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 1$</td>
</tr>
<tr>
<td>Griggs–Katona'08 $\text{ex}(n)$</td>
<td>$n \left( 1 + O\left( \frac{1}{n} \right) \right)$</td>
</tr>
</tbody>
</table>

**Conjecture**

For a fixed poset $P$

$$\text{ex}(P, n) = l(P) \binom{n}{n/2} \left( 1 + O\left( \frac{1}{n} \right) \right)$$

where $l(P)$ is the largest number of “middle” levels whose union contains no $P$. 
An explanation for the known results

Conjecture

For a fixed poset $P$

$$\text{ex}(P, n) = l(P) \binom{n}{n/2} (1 + O(1/n))$$

where $l(P)$ is the largest number of “middle” levels whose union contains no $P$. 

- Sperner'28 $\text{ex}(\), n = \binom{n}{\lfloor n/2 \rfloor}$
- Erdős'45 $\text{ex}(\), n = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}$
- Katona–Tarján'83 $\text{ex}(\), n = n \binom{n}{n/2} (1 + O(1/n))$
- Thanh'98 $\text{ex}(\), n = 2 \binom{n}{n/2} (1 + O(1/n))$
- De Bonis–Katona–Swanepoel'05 $\text{ex}(\), n = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}$
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An explanation for the known results

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For a fixed poset $P$

$$\text{ex}(P, n) = l(P) \binom{n}{n/2} (1 + O(1/n))$$

where $l(P)$ is the largest number of “middle” levels whose union contains no $P$.

**Theorem**

The conjecture is true for all the posets $P$ whose Hasse diagram is a tree. Namely, if $h(P)$ is the height of $P$ and the Hasse diagram of $P$ is a tree, then

$$\text{ex}(P, n) = (h(P) - 1) \binom{n}{n/2} (1 + O(1/n)).$$
An explanation for the known results

**Conjecture**

For a fixed poset $P$

$$\text{ex}(P, n) = l(P) \binom{n}{n/2} \left(1 + O(1/n)\right)$$

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An explanation for the known results

Conjecture

For a fixed poset $P$

$$\text{ex}(P, n) = l(P) \left( \binom{n}{n/2} \right) \left( 1 + O(1/n) \right)$$

where $l(P)$ is the largest number of “middle” levels whose union contains no $P$.

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The conjecture is true for all the posets $P$ whose Hasse diagram is a tree. Namely, if $h(P)$ is the height of $P$ and the Hasse diagram of $P$ is a tree, then

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$$\text{ex}(P, n) = (h(P) - 1) \left( \binom{n}{n/2} \right) \left( 1 + O(1/n) \right).$$
An explanation for the known results

### Conjecture

For a fixed poset $P$

$$\text{ex}(P, n) = l(P) \left( \binom{n}{n/2} \right) \left( 1 + O(1/n) \right)$$

### Theorem

The conjecture is true for all the posets $P$ whose Hasse diagram is a tree. Namely, if $h(P)$ is the height of $P$ and the Hasse diagram of $P$ is a tree, then

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Conjecture

For a fixed poset $P$

$$\text{ex}(P, n) = l(P) \binom{n}{n/2} (1 + O(1/n))$$

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The conjecture is true for all the posets $P$ whose Hasse diagram is a tree. Namely, if $h(P)$ is the height of $P$ and the Hasse diagram of $P$ is a tree, then

$$\text{ex}(P, n) = (h(P) - 1) \binom{n}{n/2} (1 + O(1/n)).$$
### Conjecture

For a fixed poset $P$

$$\text{ex}(P, n) = l(P) \left( \frac{n}{n/2} \right) \left( 1 + O(1/n) \right)$$

where $l(P)$ is the largest number of "middle" levels whose union contains no $P$.

### Theorem

The conjecture is true for all the posets $P$ whose Hasse diagram is a tree. Namely, if $h(P)$ is the height of $P$ and the Hasse diagram of $P$ is a tree, then

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An explanation for the known results

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**Theorem**

The conjecture is true for all the posets $P$ whose Hasse diagram is a tree. Namely, if $h(P)$ is the height of $P$ and the Hasse diagram of $P$ is a tree, then

$$\text{ex}(P, n) = (h(P) - 1) \binom{n}{n/2} (1 + O(1/n)).$$
An explanation for the known results

Conjecture

For a fixed poset $P$, $\text{ex}(P, n) = l(P)(n^2/2)(1 + O(1/n))$ where $l(P)$ is the largest number of "middle" levels whose union contains no $P$.

Theorem

The conjecture is true for all the posets $P$ whose Hasse diagram is a tree. Namely, if $h(P)$ is the height of $P$ and the Hasse diagram of $P$ is a tree, then

$$\text{ex}(P, n) = (h(P) - 1) \binom{n}{n/2}(1 + O(1/n))$$.
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The conjecture is true for all the posets $P$ whose Hasse diagram is a tree. Namely, if $h(P)$ is the height of $P$ and the Hasse diagram of $P$ is a tree, then

$$\text{ex}(P, n) = (h(P) - 1) \binom{n}{n/2} (1 + O(1/n)).$$

Qualitatively explains all the previously known result including $\bullet \subset$ poset because $\bullet \subset \bowtie$.

For $h(P) = 2$ independently proved by Griggs and Lu. They also proved the conjecture for a large class of posets with $h(P) = 2$, whose Hasse diagram is not a tree.
First idea for $h(P) = 2$

Have: $|\mathcal{F}| \geq (1 + \varepsilon)\binom{n}{n/2}$, poset $P = \mathbb{N}$
Want: an embedding of $P$ into poset $(\mathcal{F}, \subset)$

Idea

_Treat poset $(\mathcal{F}, \subset)$ as a graph, and embed tree into it._
First idea for $h(P) = 2$

Have: $|\mathcal{F}| \geq (1 + \varepsilon) \binom{n}{n/2}$, poset $P = \mathbb{N}$

Want: an embedding of $P$ into poset $(\mathcal{F}, \subset)$

Idea

_Treat poset $(\mathcal{F}, \subset)$ as a graph, and embed tree into it._

- Graph $G$. Vertex set $V(G) = \mathcal{F}$, for $F_1, F_2 \in \mathcal{F}$ edge $F_1 \sim F_2$ if either $F_1 \subset F_2$ or $F_2 \subset F_1$.
- If $\varepsilon > 0$ the average degree of $G$ is at least 100.
- Subgraph $G'$ in which minimum degree is at least 50. Will embed into $G'$.
First idea for $h(P) = 2$

Have: $|\mathcal{F}| \geq (1 + \varepsilon) \binom{n}{n/2}$, poset $P = \mathbb{N}$

Want: an embedding of $P$ into poset $(\mathcal{F}, \subset)$

**Idea**

*Treat poset $(\mathcal{F}, \subset)$ as a graph, and embed tree into it.*

- Graph $G$. Vertex set $V(G) = \mathcal{F}$, for $F_1, F_2 \in \mathcal{F}$ edge $F_1 \sim F_2$ if either $F_1 \subset F_2$ or $F_2 \subset F_1$.
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First idea for $h(P) = 2$

Have: $|\mathcal{F}| \geq (1 + \varepsilon)\left(\frac{n}{2}\right)$, poset $P = \mathbb{N}$

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To embed: 

```
\begin{tikzpicture}
  \node (a) at (0,0) {a};
  \node (b) at (1,1) {b};
  \node (c) at (1,-1) {c};
  \node (d) at (2,0) {d};
  \draw (a) -- (b);
  \draw (a) -- (c);
\end{tikzpicture}
```

Having embedded:

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To embed: $\begin{array}{ccc} a & b & d \\ \bullet & \bullet & \bullet \end{array}$

Having embedded: $\begin{array}{ccc} a & c \end{array}$

Done!
First idea for \( h(P) = 2 \)

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To embed: 

Having embedded: 

Done next step
Trouble with $h(P) \geq 3$

Difficulty
For $h(P) \geq 3$ instead of graphs have to use $h(P)$-uniform hypergraphs, and there is no good analogue of minimum degree.

Problem
How to embed a tree into a graph of large average degree without using minimum degree?
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*How to embed a tree into a graph of large average degree without using minimum degree?*

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*For every tree \( T \) there is a \( d = d(T) \) such that every graph \( G \) of average degree \( \geq d \) contains \( T \).*
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<table>
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**Proof.**

Induction on $|T|$. If $|T| = 1$, trivial. Let $v$ be a leaf. Else let $T' = T \setminus \{v\}$, and $d(T) = 2d(T') + 4|T|

- Let $V' = \{x \in V(G) : \deg(x) \geq d(T)/4\}$. Define $G' = G|_{V'}$

- Average degree of $G'$ is at least $d(T)/2$.

- Find an embedding of $T'$ into $G'$.

- Since $\deg(u) \geq |T|$ in $G$, extend the embedding.