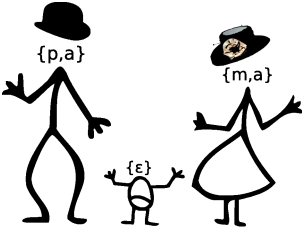
Set families with a forbidden subposet

Boris Bukh

August 2008



Introduction

$$[n] = \{1, \ldots, n\} \qquad 2^{[n]} = \{F : F \subset [n]\}$$

Definition

Set family is a collection of subsets of $[n]$. In symbols, $\mathcal{F} \subset 2^{[n]}$.

Theorem (Sperner'28)

Suppose $\mathcal{F} \subset 2^{[n]}$ is a set family such that for no distinct $F_1, F_2 \subset \mathcal{F}$ the inclusion $F_1 \subset F_2$ holds. Then

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

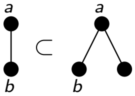
Equality is attained for $\mathcal{F} = \binom{[n]}{\lfloor n/2 \rfloor} = \{F \subset [n] : |F| = \lfloor n/2 \rfloor\}$.

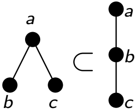
Reformulation

Definition

P_1 is a subposet of P_2 (written $P_1 \subset P_2$) if there is an injective $f: P_1 \to P_2$ such that

$$x <_{P_1} y \implies f(x) <_{P_2} f(y)$$

Examples: 

and 

Way of thinking

A set family is a poset under inclusion. For $F_1, F_2 \in \mathcal{F}$ set $F_1 \leq_{\mathcal{F}} F_2$ if $F_1 \subset F_2$.

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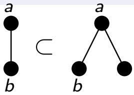
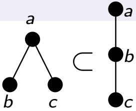
Examples: 

and 

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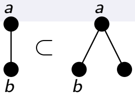
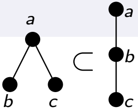
Examples: 

and 

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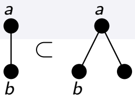
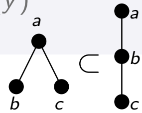
Examples: 

and 

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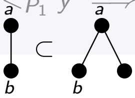
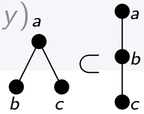
Examples: 

and 

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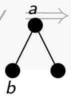
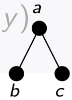
$\subset$ 

and 

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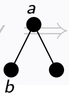
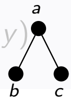
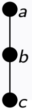
$\subset$ 

and 

$\subset$ 

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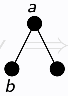
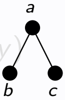
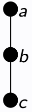
$\subset$ 

and 

$\subset$ 

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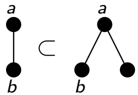
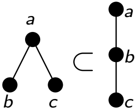
Examples: 

and 

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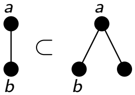
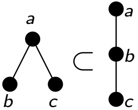
Examples: 

and 

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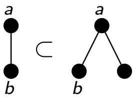
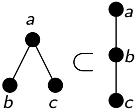
Examples: 

and 

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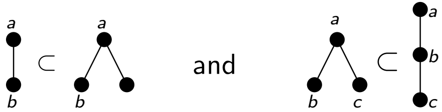
Examples: 

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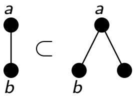
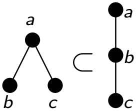
Examples: 

and 

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Theorem (Sperner'28)

 $\not\subset \mathcal{F} \implies |\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$

Reformulation and the known results

Notation

Largest family in $[n]$ not containing poset P has size

$$\mathrm{ex}(P,n) = \max_{\substack{P \not\subset \mathcal{F} \\ \mathcal{F} \subset 2^{[n]}}} |\mathcal{F}|$$

Sperner'28 — $\mathrm{ex}($$, n) = \binom{n}{\lfloor n/2 \rfloor}$

Erdős'45 — $\mathrm{ex}($$, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}$

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Thanh'98	$\mathrm{ex}(\pitchfork, n) = 2\binom{n}{n/2}\left(1 + O(1/n)\right)$
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$$\mathrm{ex}(P, n) = l(P)\binom{n}{n/2}\left(1 + O(1/n)\right)$$

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Theorem

The conjecture is true for all the posets P whose Hasse diagram is a tree. Namely, if $h(P)$ is the height of P and the Hasse diagram of P is a tree, then

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Qualitatively explains all the previously known result including $\bowtie$ poset because $\bowtie \subset$ .

For $h(P) = 2$ independently proved by Griggs and Lu. They also proved the conjecture for a large class of posets with $h(P) = 2$, whose Hasse diagram is not a tree.

First idea for $h(P) = 2$

Have: $|\mathcal{F}| \geq (1+\varepsilon)\binom{n}{n/2}$, poset $P = N$
Want: an embedding of P into poset $(\mathcal{F}, \subset)$

Idea

Treat poset $(\mathcal{F}, \subset)$ as a graph, and embed tree into it.

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- Graph G. Vertex set $V(G) = \mathcal{F}$, for $F_1, F_2 \in \mathcal{F}$ edge $F_1 \sim F_2$ if either $F_1 \subset F_2$ or $F_2 \subset F_1$.
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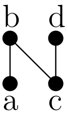
To embed: 

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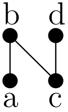
To embed: 

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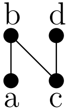
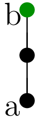
To embed: 

Having embedded: 

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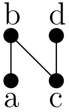
To embed: 

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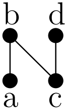
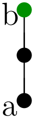
To embed: 

Having embedded: 

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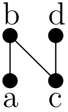
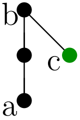
To embed: 

Having embedded: 

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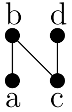
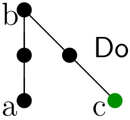
Idea

Treat poset $(\mathcal{F}, \subset)$ as a graph, and embed tree into it.

- Graph G. Vertex set $V(G) = \mathcal{F}$, for $F_1, F_2 \in \mathcal{F}$ edge $F_1 \sim F_2$ if either $F_1 \subset F_2$ or $F_2 \subset F_1$.
- If $\varepsilon > 0$ the average degree of G is at least 100.
- Subgraph G' in which minimum degree is at least 50. Will embed into G'

To embed: 

Having embedded: 

Done next step

Trouble with $h(P) \geq 3$

Difficulty

For $h(P) \geq 3$ instead of graphs have to use $h(P)$-uniform hypergraphs, and there is no good analogue of minimum degree.

Problem

How to embed a tree into a graph of large average degree without using minimum degree?

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Theorem

For every tree T there is a $d = d(T)$ such that every graph G of average degree $\geq d$ contains T.

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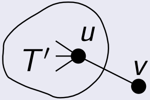
Proof.

Induction on $|T|$. If $|T| = 1$, trivial. Let v be a leaf. Else let $T' = T \setminus \{v\}$, and $d(T) = 2d(T') + 4|T|$

- Let $V' = \{x \in V(G) : \deg(x) \geq d(T)/4\}$. Define $G' = G|_{V'}$
- Average degree of G' is at least $d(T)/2$.
- Find an embedding of T' into G'.
- Since $\deg(u) \geq |T|$ in G, extend the embedding.



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