Set families with a forbidden subposet

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Introduction

$$[n] = \{1, \dots, n\} \qquad 2^{[n]} = \{F : F \subset [n]\}$$

Definition

Set family is a collection of subsets of [n]. In symbols, $\mathcal{F} \subset 2^{[n]}$.

Theorem (Sperner'28)

Suppose $\mathcal{F} \subset 2^{[n]}$ is a set family such that for no distinct $F_1, F_2 \subset \mathcal{F}$ the inclusion $F_1 \subset F_2$ holds. Then

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Equality is attained for $\mathcal{F} = {[n] \choose \lfloor n/2 \rfloor} = \{F \subset [n] : |F| = \lfloor n/2 \rfloor\}.$

 P_1 is a subposet of P_2 (written $P_1 \subset P_2$) if there is an injective $f: P_1 \rightarrow P_2$ such that

$$x <_{P_1} y \implies f(x) <_{P_2} f(y)$$

Examples:





Way of thinking

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A set family is a poset under inclusion. For $F_1, F_2 \in \mathcal{F}$ set $F_1 \leq_{\mathcal{F}} F_2$ if $F_1 \subset F_2$.

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A set family is a poset under inclusion. For $F_1, F_2 \in \mathcal{F}$ set $F_1 \leq_{\mathcal{F}} F_2$ if $F_1 \subset F_2$.

Theorem (Sperner'28)

$$\mathbf{I} \not\subset \mathcal{F} \implies |\mathcal{F}| \le \binom{n}{\lfloor n/2 \rfloor}$$

Largest family in [n] not containing poset P has size

$$\exp(P,n) = \max_{\substack{P
ot \in \mathcal{F} \ \mathcal{F} \subset 2^{[n]}}} |\mathcal{F}|$$

$$ex(l, n) = {n \choose \lfloor n/2 \rfloor}$$

$$ex(l, n) = {n \choose \lfloor n/2 \rfloor} + {n \choose \lfloor n/2 \rfloor + 1}$$

$$ex(V, n) = {n \choose n/2} (1 + O(1/n))$$

$$ex(k, n) = 2{n \choose n/2} (1 + O(1/n))$$

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Largest family in [n] not containing poset P has size

$$\begin{split} & \max_{\substack{P \not\in \mathcal{F} \\ \mathcal{F} \notin \mathcal{F}}} |\mathcal{F}| \\ & \mathcal{F} e \underline{x} (\underline{i}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} \\ & e x (\underline{i}, n) = \binom{n}{n/2} (1 + O(1/n)) \\ & e x (\underline{i}, n) = 2\binom{n}{n/2} (1 + O(1/n)) \\ & e x (\underline{i}, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} \\ & e x (\mathbb{N}, n) = \binom{n}{n/2} (1 + O(1/n)) \end{split}$$

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$$\begin{array}{l} \max \left| \begin{array}{c} \mathcal{F} \\ \mathcal{Pex}(1,n) = \binom{n}{\lfloor n/2 \rfloor} \\ \mathcal{F} \\ \mathcal{C} \\ \mathcal{C$$

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$$= \operatorname{rex}(1, \overline{n}) = \binom{n}{\lfloor n/2 \rfloor}$$

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Largest family in [n] not containing poset P has size

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Conjecture

For a fixed poset P

$$ex(P, n) = I(P)\binom{n}{n/2}(1 + O(1/n))$$

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Katona–Tarján'83	$ex(\mathbb{V},n) = \binom{n}{n/2} \left(1 + O(1/n)\right)$
Thanh'98	$\exp(\frac{1}{n},n)=2\binom{n}{n/2}\left(1+O(1/n)\right)$
De Bonis–Katona–Swanepoel'05	$ex(\aleph, n) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}$
Griggs-Katona'08	ex(N, n) = (n, n)(1 + O(1/n))
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where I(P) is the largest number of "middle" levels whose union contains no P.

Theorem

$$ex(P, n) = (h(P) - 1) {n \choose n/2} (1 + O(1/n)).$$

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The conjecture is true for all the posets P whose Hasse diagram is a tree. Namely, if h(P) is the height of P and the Hasse diagram of P is a tree, then

$$ex(P, n) = (h(P) - 1) {n \choose n/2} (1 + O(1/n)).$$

Qualitatively explains all the previously known result including \mathbb{M} poset because $\mathbb{M} \subset X$. For h(P) = 2 independently proved by Griggs and Lu. They also

proved the conjecture for a large class of posets with h(P) = 2, whose Hasse diagram is not a tree.

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Idea

- Graph G. Vertex set $V(G) = \mathcal{F}$, for $F_1, F_2 \in \mathcal{F}$ edge $F_1 \sim F_2$ if either $F_1 \subset F_2$ or $F_2 \subset F_1$.
- If $\varepsilon > 0$ the average degree of G is at least 100.
- Subgraph G' in which minimum degree is at least 50. Will embed into G'

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Idea

Treat poset (\mathcal{F}, \subset) as a graph, and embed tree into it.

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Having embedded:

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How to embed a tree into a graph of large average degree without using minimum degree?

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Proof.

Induction on |T|. If |T| = 1, trivial. Let v be a leaf. Else let $T' = T \setminus \{v\}$, and d(T) = 2d(T') + 4|T|

- Let $V' = \{x \in V(G) : \deg(x) \ge d(T)/4\}$. Define $G' = G|_{V'}$
- Average degree of G' is at least d(T)/2.
- Find an embedding of T' into G'.

