Sum-product estimate for $|A+A|+|f(A)+B|$ for a quadratic polynomial f

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Theorem 1. *Let $f \in \mathbb{F}_p[X]$ be a polynomial of degree 2. Then for all sets $A, B \subset \mathbb{F}_p$ satisfying $|A|, |B| \le \sqrt{p}$, we have*

$$|A+A| + |f(A)+B| \gg |A||B|^{1/180}.$$

Lemma 2 (A variation on [Gar08])**.** *Suppose $A, B \subset \mathbb{F}_p$ and $G \subset A \times B$ is a bipartite graph with at least $\frac{1}{K}|A||B|$ edges. Then*

$$|A+A| + |A \cdot_G B| \gg \frac{\min(|B|, p/|A|)^{1/25-o(1)}}{K}|A|.$$

Corollary 3. *Suppose sets $A, B \subset \mathbb{F}_p$ and a graph $G \subset A \times B$ satisfy $|A|, |B| \le p^{(1+1/29)/2}$ and $|G| \ge \frac{1}{K}|A||B|$. Then*

$$|A+A| + |A \cdot_G B| \gg \frac{|B|^{1/29}}{K}|A|.$$

Theorem 1 from [Gar08] is a special case of Lemma 2 for $G = A \times B$. However, the proof from [Gar08] carries almost verbatim to establish the stronger result. Below we outline the changes to that argument to the case when $G \neq A \times B$.

Proof sketch for Lemma 2. Let $\lambda \cdot A = \{\lambda a : a \in A\}$ denote the dilate of A by λ. For $b \in B$ put $A_b = \{a : (a,b) \in G\}$. Suppose $|A+A| + |A \cdot_G B| \le \Delta |A|$. Since $\sum_{b,b' \in B} |b \cdot A_b \cap b' \cdot A_{b'}| \ge |G|^2/|A \cdot_G B| \ge |A||B|^2/\Delta K^2$, there is a fixed $b_0 \in B$ for which $\sum_{b \in B} |b \cdot A_b \cap b_0 \cdot A_{b_0}| \ge |A||B|/\Delta K^2$. Define

$$B_1 = \{b \in B : |b \cdot A_b \cap b_0 \cdot A_{b_0}| \ge \frac{|A|}{2\Delta K^2}\}.$$

For $b \in B_1$ from the Ruzsa triangle inequalities we deduce

$$\begin{aligned} |b \cdot A \pm b_0 \cdot A| &\le \frac{|b \cdot A + (b \cdot A_b \cap b_0 \cdot A_{b_0})||(b \cdot A_b \cap b_0 \cdot A_{b_0}) + b_0 \cdot A|}{|b \cdot A_b \cap b_0 \cdot A_{b_0}|} \\ &\le \frac{|A+A|^2}{|A|/2\Delta K^2} \le 2|A|\Delta^3 K^2. \end{aligned}$$

For a given $a \in A$, let $B_1(a) = \{b : (a,b) \in G,\ b \in B_1,\ ab \in b_0 \cdot A_{b_0}\}$. Then

$$\sum_{a\in A} |B_1(a)| = \sum_{b\in B_1} |b \cdot A_b \cap b_0 \cdot A_{b_0}| \geq \frac{|A||B|}{2K^2\Delta}.$$

Replacing the $0.5|X|^2|Y|/|XY|$ by $|X|^2|Y|/K|X \cdot_G Y|$ in Lemma 3 of [Gar08], and carrying the rest of the proof as in [Gar08], then after simple, but tedious calculations we obtain

$$\Delta \gg \min(|B|^{1/8-o(1)}/K, (|B|/K^{17})^{1/25}, (p/K^{16}|A|)^{1/25-o(1)}) \geq \min(|B|, p/|A|)^{1/25-o(1)}/K.$$

□

Proof of Theorem 1. Since $|f(A) + B| \geq |B|$, it suffices to deal only with the case $|B| \leq |A|^2$. Suppose $f(x) = \alpha x^2 + \beta x + \gamma$ is a quadratic polynomial with $\alpha \neq 0$. Assume that $|A + A| + |f(A) + B| \leq \Delta|A|$. From the Cauchy–Schwarz inequality it follows that there are at least $(|A||B|)^2/|f(A) + B| \geq |A||B|^2\Delta^{-1}$ solutions to

$$\alpha a_1^2 + \beta a_1 + \gamma + b_1 = \alpha a_2^2 + \beta a_2 + \gamma + b_2, \qquad a_1, a_2 \in A,\ b_1, b_2 \in B.$$

Since $a_1^2 - a_2^2 = (a_1 - a_2)(a_1 + a_2)$ after rearranging the terms we conclude that there are at least $\Delta^{-1}|A||B|^2$ solutions to

$$s \cdot \frac{1}{b_2 - b_1} = d, \qquad s \in \alpha \cdot (A + A) + \beta,\ d \in (A - A)^{-1},\ b_1, b_2 \in B. \tag{1}$$

For $(b_1, b_2) \in B \times B$ we let $r(b_1, b_2)$ be the number ways to extend it to a solution of (1). Let $G = \{(b_1, b_2) \in B \times B : r(b_1, b_2) \geq \frac{1}{2}\Delta^{-1}|A|\}$. Since the number of solutions to (1) with $(b_1, b_2) \notin G$ is at most $|B|^2 \cdot \frac{1}{2}\Delta^{-1}|A|$, the number of solutions to the equation subject to $(b_1, b_2) \in G$ is at least $\frac{1}{2}\Delta^{-1}|A||B|^2$. Since we always have $r(b_1, b_2) \leq |A + A| \leq \Delta|A|$, it follows that $|G| \geq \frac{1}{2}|B|^2\Delta^{-2}$. For $c \in B - B$ let $R(c)$ be the number of solutions to $c = b_2 - b_1$ with $(b_1, b_2) \in G$. Let $G_i = \{(b_1, b_2) \in G : 2^{i-1} \leq R(b_2 - b_1) \leq 2^i\}$. Each $(b_1, b_2) \in G$ belongs to one of $G_1, \ldots, G_{\log|B|}$. Pick an i so that $|G_i| \geq |G|/\log|B|$. Let $C = \{\frac{1}{b_2 - b_1} : (b_1, b_2) \in G_i\}$. Note that $2^{-i}|G_i| \leq |C| \leq 2^{1-i}|G_i|$. Each solution to

$$sc = d, \qquad s \in \alpha \cdot (A + A) + \beta,\ d \in (A - A)^{-1},\ c \in C \tag{2}$$

gives rise to $R(c) \leq 2^i$ solutions to (1). Thus the number of solutions to (2) is at least

$$2^{-i}\big(|G_i| \cdot \tfrac{1}{2}\Delta^{-1}|A|\big) \geq (\tfrac{1}{2}|C|) \cdot \tfrac{1}{2}\Delta^{-1}|A| = \tfrac{1}{4}\Delta^{-1}|A||C| \geq \tfrac{1}{4}\Delta^{-2}|A + A||C|$$

By Plünnecke's inequality $|(A + A) + (A + A)| \leq \Delta^4|A|$ and by the triangle inequality $|A - A| \leq \Delta^2|A|$. If $|C| \geq p^{(1+1/29)/2}$, then $|C| \leq |A - A| \leq \Delta^2|A|$ implies that $\Delta \geq |A|^{1/58} \geq |B|^{1/116}$. Thus $|C| \leq p^{(1+1/29)/2}$ with a similar bound for $|A + A|$.

Let $H = \{(s, c) \in (\alpha \cdot (A + A) + \beta) \times C : sc \in (A - A)^{-1}\}$. Corollary 3 with $K = 4\Delta^2$ applied to

$$|(\alpha \cdot (A + A) + \beta) + (\alpha \cdot (A + A) + \beta)| + |(\alpha \cdot (A + A) + \beta) \cdot_H C|$$

yields $|C|^{1/29}/\Delta^2 \leq \Delta^4$. Therefore

$$\Delta^{174} \geq |C| \geq \frac{|G_i|}{|B|} \geq \frac{|G|}{|B|\log|B|} \geq \frac{|B|}{2\Delta^2\log|B|},$$

implying $\Delta \geq |B|^{1/176-o(1)}$. □

References

[Gar08] M. Z. Garaev. A quantified version of Bourgain's sum-product estimate in $\mathbb{F}_p$ for subsets of incomparable sizes. *Electron. J. Combin.*, 15(1):Research paper 58, 8, 2008.