THE CLASSIFICATION OF THE ORDER INDISCERNIBLES OF REAL CLOSED FIELDS AND OTHER THEORIES

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The primary result is a complete description of the types of infinite indiscernible sequences in the theory of real closed fields. There are two important parameters in the description. One indicates the Dedekind cut of the type in the rationals and the other specifies a particular exponent in determining how far apart the indiscernibles are. For any choice of these two parameters there are less than twenty five types. This information is used to describe when the types are definable from each other. The relationship between the indiscernibility types and the classical description of real closed fields is also given.

Another important example which is analyzed is the theory of divisible ordered abelian groups. A complete classification of the indiscernibility types is given as well as the relationship of the indiscernibility types to the models of the theory. For this theory we give the classification of the indiscernibility types over arbitrary sets of constants.

Several other examples are also included.

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1. INTRODUCTION

During the last forty years there has been a significant effort in applying results in model theory to algebra. In the last few years there has been an increasing interest in using the newer techniques of model theory to classify the models of a theory. This thesis will describe what the order indiscernibles look like in some particular theories. The original use of order indiscernibles dates back to the 1950's work of Ehrenfeucht and Mostowski. They were interested in using order indiscernibles to build models with various kinds of automorphism groups. In the 1960's Morley made use of indiscernibles to obtain some important model theoretic results. More recently, Shelah has used them to show that there are a large number of models for unstable theories. Shelah has also used pure indiscernibles in classifying models of stable theories. And of course, order indiscernibles have also been used by many other researchers.

The focus of this thesis is on unstable theories and hence here we cannot use the tools of stability theory. Yet the idea of using indiscernibles to describe models is an important part of stability theory. So the success of applications of indiscernibles to unstable theories is a hopeful signal that finer structure theory can also be obtained for stable theories.

We begin in Chapter 2, with a look at the order indiscernibles for divisible ordered abelian groups. In Chapter 3, we go to the more complicated example of real closed fields. Finally in Chapter 4, we present a few more examples to get a more complete view.

Definitions and Notations

Definition: Let $\mathcal{A}$ be a model for the language L, and let C be a subset of A which carries a relation < that simply orders C. (Note that < need not be a relation in L.) We say that C is a set of order indiscernibles in $\mathcal{A}$ with respect to < iff for all n, and for all finite sequences $c_1 < c_2 \ldots c_n$ and $d_1 < d_2 \ldots d_n$ from C, and all formulas Θ, $\mathcal{A} \models \Theta(c_1, c_2, \ldots c_n) \leftrightarrow \Theta(d_1, d_2, \ldots d_n)$. (That is $\Theta(c_1 \ldots c_n)$ is true in $\mathcal{A}$ iff $\Theta(d_1 \ldots d_n)$ is true in $\mathcal{A}$.) Sometimes we will simply say that C is an indiscernible set (in $\mathcal{A}$).

Definition: A model $\mathcal{A}$ is a real closed field iff $\mathcal{A}$ is a model of Th(R,+,·,0,1) iff $\mathcal{A}$ satisfies the field axioms and

1) $\forall x \exists y (y^2 = x \vee y^2 + x = 0)$

$2)_n$ $x_o^2 + x_1^2 + \ldots + x_n^2 = 0 \rightarrow x_o = 0 \quad \forall n \in \omega$

$3)_n$ $x_n \neq 0 \rightarrow \exists y (x_n y^n + x_{n-1} y^{n-1} + \ldots x_1 y + x_o = 0)$ for all odd $n \in \omega$.

If $\mathcal{A}$ is a real closed field we introduce < into the language and define it by $x<y \leftrightarrow \exists z : z^2 = y - x$. This makes < a linear order on $\mathcal{A}$.

Definition: A model <A, <, +, 0> is a divisible ordered abelian group iff <A, +, 0> is a divisible abelian group which is linearly ordered by <.

We will abbreviate Divisible Ordered Abelian Group by DOAG.

Definition: A theory T has quantifier elimination iff every first order formula is equivalent to a formula with no quantifiers. (Abbreviated as T has QE.)

If a theory has QE then the definitions for indiscernibles may be simplified.

Proposition 1.1: A subset C of a real closed field $\mathcal{A}$ is a set of order indiscernibles iff for any n and any polynomial $p(x_1,\ldots x_n)$ over the rationals and any $c_1 < c_2 < \ldots < c_n$ and $d_1 < d_2 < \ldots < d_n$ from C, $\models p(c_1,c_2,\ldots c_n) > 0 \leftrightarrow p(d_1,d_2,\ldots d_n) > 0$.

Proof: Just use the fact that RCF has QE. □

Remark 1.2: Technically C may be ordered by $\lessdot$ outside of L, but since $\mathcal{A}$ is linearly ordered $\mathcal{A}$ is either < or > of L . So we may assume $\lessdot$ is <.

Remark 1.3: Since RCF has QE we may simply write $p(c_1\ldots c_n) > 0$ instead of $\mathcal{A} \models p(c_1,c_2,\ldots c_n) > 0$. This is because if $p(c_1\ldots c_n) > 0$ is true in $\mathcal{A}$ then it is true in any model of RCF containing $c_1\ldots c_n$.

Remark 1.4: Technically the language of RCF does not include subtraction, division or rational numbers. But each of these is definable in L, so we may just as well use them. Note that rational powers are also definable.

Proposition 1.5: A subset C of a divisible ordered abelian group A is a set of order indiscernibles iff for every integer n and for every $c_1 < c_2 < \ldots < c_n$ and $d_1 < d_2 < \ldots < d_n$ in C, $A \models \sum n_i c_i > 0 \leftrightarrow \sum n_i d_i > 0$.

Proof: DOAG has QE. □

Remark 1.6: Remarks 1.2-1.4 also hold for DOAG's, except that division and rational powers are not definable.

We may sometimes write a sequence $c_1 < c_2 < \ldots < c_n < \ldots$ as $\langle c_i : i \in N^+ \rangle$. More generally, if $c_i < c_j$ whenever $i < j$ we will write the sequence as $\langle c_i : i \in I \rangle$. Sometimes we will just write $\langle c_i \rangle$ if the index set is obvious. The concatenation of the sequences $\langle c_i : i \in I \rangle$ and $\langle d_j : j \in J \rangle$ when $c_i < d_j$ for all $i \in I$ and $j \in J$, is written as $\langle c_i : i \in I, d_j : j \in J \rangle$. If we want the index set I in the opposite order we will write I^*.

Definition: A 1-type τ for the theory T is a maximal collection of formulas $\{\Theta_k(x)\}$ which is satisfied by an element of some model of T.

Definition: An (infinite) indiscernible type τ for the theory T is a maximal collection of formulas $\{\Theta_k(x_1,\ldots x_n):k,n\in N^+\}$ which is satisfied by an indiscernible sequence $<c_i:i\in N^+>$ of a model of T.

Definition: An indiscernible type τ of index set I for the theory T is a maximal collection of formulas $\{\Theta_k(x_{i_1},\ldots x_{i_n}):k,n\in N^+, i_1,\ldots i_n\in I\}$ which is satisfied by an indiscernible sequence $<c_i:i\in I>$ of a model of T.
The use of τ's of index set other than N^+ is merely a technical device.

Proposition 1.7: There is a 1 to 1 correspondence of indiscernibility types to indiscernibility types of any infinite index set I. It is given by: the indiscernible type τ corresponds to $\tau_I=\{\Theta(x_{i_1},\ldots x_{i_n}):i_1<i_2\ldots i_n$ in I and $\Theta(x_1,x_2\ldots x_n)\in\tau\}$.

Proof: By the completeness theorem and the indiscernibility of τ, τ_I is consistent. Since τ is maximal so is τ_I. Finally, every indiscernibility type of index set I must be some τ_I. □

One of the main goals in the following chapters will be classifications of the indiscernibility types of same T. But sometimes we will just describe a τ as a τ_I for some other index set I. Also if some collection of formulas $\{\Theta_k(x_1\ldots x_n):n,k\in N^+\}$

has a unique extension to an indiscernibility type, we may use it to represent the indiscernibility type. In particular, if T has QE we only need to look at collections of atomic formulas.

2. DIVISIBLE ORDERED ABELIAN GROUPS

The primary result in this chapter will be a classification of the types of indiscernibles for DOAG's. We will also show how they relate to each other, how they describe models, and a little about how they relate to automorphisms or models. Finally we will classify the indiscernibles over sets of constants. Later, in Chapter 3, we will see how they relate to the indiscernibles of RCF's.

A. Classification of the Indiscernibility Types

Theorem 2.1: There are exactly 3 types of indiscernibles with $x_1>0$. They are determined by:

1) $2x_1<x_2$

2) $x_2<2x_1$ and $2(x_2-x_1)<(x_3-x_1)$

3) $x_2<2x_1$ and $(x_3-x_1)<2(x_2-x_1)$

Proof: These three cases are a partition of the types, so it suffices to show that there is exactly one type for each case. We will first show uniqueness and then existence.

(1) Uniqueness: Suppose an indiscernibility type τ contains $x_2>(1+\varepsilon)x_1$ for some $\varepsilon\in Q^+$. (Recall that we may use rational coefficients since they are definable in the language.) Then $x_3>(1+\varepsilon)x_2\in\tau$ by indiscernibility. Also $(1+\varepsilon)x_2>(1+\varepsilon)^2 x_1\in\tau$ by multiplying the hypothesis by $(1+\varepsilon)$. Hence $x_3>(1+\varepsilon)^2x_1\in\tau$ by the transitivity of <. So again by indiscernibility we have $x_2>(1+\varepsilon)^2x_1\in\tau$. Iterating this procedure we get $x_2>(1+\varepsilon)^n x_1\in\tau$ for all n. Hence $x_2>Hx_1\in\tau$ for all $H\in Q^+$. In particular, since our hypothesis has $2x_1<x_2\in\tau$ we have $x_2>Hx_1\in\tau$ for all $H\in Q^+$. We also have $x_1>0$. Now any nonzero term is of the form $a_nx_n+a_{n-1}x_{n-1}+\ldots+a_1x_1$ where $a_n\neq 0$. We want to determine when a term is > or <0. We have, $x_n>Hx_{n-1}$, $x_n>Hx_{n-2},\ldots,x_n>Hx_1$ are in τ for all H>0 by $x_2>Hx_1\in\tau$ and indiscernibility. So if $a_n>0$ then $\frac{a_n}{2}x_n>a_{n-1}x_{n-1}+\ldots a_1x_1\in\tau$. Hence, $a_nx_n+a_{n-1}x_{n-1}+\ldots a_1x_1 > \frac{a_n}{2}x_n>0\in\tau$ (i.e., τ would have the term >0). If $a_n<0$ a similar argument would show that the term is <0 in τ. Hence all the atomic formulas are determined. Thus by QE, there is a unique indiscernibility type for case (1).

(1) Existence: By direct construction, or by the compactness theorem, we can find c_i's in a DOAG such that $c_{n+1}>Hc_n$ and $c_n>0$ for all $H\in Q^+$ and for all $n\in N^+$. Suppose we choose indices $i_1<i_2\ldots<i_n$ and $j_1<j_2\ldots<j_n$ and $a_n\neq 0$. Then $a_nc_{i_n}+\ldots a_1c_{i_1}>0$ iff $a_n>0$. (By the same argument as in the Uniqueness part.)

So $a_n c_{i_n} + \ldots a_1 c_{i_1} > 0$ iff $a_n > 0$ iff $a_n c_{j_n} + \ldots a_1 c_{j_1} > 0$. Hence the c_i's are a set of indiscernibles. Furthermore, they satisfy $x_2 > 2x_1$ and $x_1 > 0$. So they are type (1) indiscernibles and we have existence.

This shows case (1) and we now show case (2).

(2) Uniqueness: Let $\langle c_i \rangle$ be any sequence of indiscernibles satisfying $x_1 > 0, x_2 < 2x_1$, and $2(x_2 - x_1) < (x_3 - x_1)$. Then by the argument in (1) Uniqueness, we must have $c_2 < (1+\varepsilon)c_1$ for all $\varepsilon \in Q^+$. Let $d_i = c_i - c_1$. Then $2d_i < d_j$ for $i<j$, by hypothesis and indiscernibility. So by part (1) the d_i's are type (1) indiscernibles. Also, for $H>0$, $Hd_n = H(c_n - c_1) < c_1 \leftrightarrow c_n - c_1 < \frac{1}{H} c_1 \leftrightarrow c_n < c_1 + \frac{1}{H} c_1 \leftrightarrow c_n < (1+\frac{1}{H})c_1 \leftrightarrow$ Truth (by our hypothesis). So for all $H \in Q^+$ and $n>1$, $Hd_n < c_1$. Hence by the argument of part (1) Existence, $\langle d_i : i \in N^+ \setminus \{1\}, c_1 \rangle$ is a sequence of case (1) indiscernibles. (Note that the index set is not isomorphic to N^+, but the same argument works since a formula only uses a finite number of parameters from the sequence.) Now any term $a_n c_n + \ldots a_1 c_1 =$ $a_n(c_n - c_1) + \ldots + a_2(c_2 - c_1) + bc_1$ (where $b = a_1 + a_2 + a_3 \ldots + a_n$) $=$ $a_n d_n + \ldots a_2 d_2 + bc_1 = bc_1 + a_n d_n + \ldots + a_2 d_2$. By the (1) indiscernibility of $\langle d_i, c_1 \rangle$ the sign of this term is determined. Hence C must satisfy $a_n x_n + \ldots a_1 x_1 \gtrless 0$ depending on the sign of the first nonzero coefficient of $bx_1 + a_n(x_n - x_1) + \ldots a_2(x_2 - x_1)$. Since the choice of C was arbitrary, all indiscernible types

extending $\{x_1>0, x_2<2x_1, 2(x_2-x_1)<(x_3-x_1)\}$ must satisfy the same formulas as above. Thus, since all atomic formulas are determined, there is a unique indiscernibility type for case (2).

(2) Existence: Let $<d_i : i\in N^+, d_\omega>$ be a sequence of case (1) indiscernibles. (By (1) we have the existence of indiscernibles of order type N^+ and so by the compactness theorem we have the existence of (1) indiscernibles for any infinite index set.) Now let $c_i=d_i+d_\omega$. (Note that we are simply reversing the coding given in (2) Uniqueness.) By the indiscernibility of the d_i's, the c_i's are also indiscernibles. We have $2c_1=2(d_1+d_\omega) = 2d_1+2d_\omega>d_2+d_\omega=c_2$. Also, $2(c_2-c_1)=2(d_2+d_\omega)-(d_1+d_\omega)) = 2(d_2-d_1)<d_3-d_1=c_3-c_1$. So the c_i's satisfy the formulas for case (2).

(3) Uniqueness: Choose any indiscernible sequence $<c_i : i\in N^+, c_\omega>$ satisfying the case (3) formulas. As in part (2) we have $c_2<(1+\varepsilon)c_1$ for all $\varepsilon\in Q^+$. Let $d_i=c_\omega-c_i$ for $i\in N^+$. Then $c_3-c_1<2(c_2-c_1)$ by hypothesis so $c_3-c_2<c_2-c_1$ by subtracting c_2-c_1 from both sides. Thus $2(c_3-c_2)<(c_3-c_1)$. So $2(c_\omega-c_2)<(c_\omega-c_1)$ by indiscernibility. Hence $2d_2<d_1$. The d_i's are indiscernibles since the c_i's are indiscernibles. Hence the d_i's are in fact type (1) indiscernibles with index set $(N^+)^*$. (Note that the ordering on the d_i's is reversed.)

Also $Hd_2=H(c_\omega-c_2)<c_2$ for all $H\in Q^+$ (since $c_\omega<(1+\varepsilon)c_2$). So $<d_1:i\in(N^+)^*,c_1>$ is a sequence of (1) indiscernibles. As in case (2), the c_i's are definable from this sequence and hence all term inequalities reduce to questions about $<d_i$'s$,c_1>$. And these are determined independent of the particular choice of C. Thus there is a unique type extending the conditions of case (3). (Technically we have only shown that there exists a unique case (3) type of index set $<N^+,\{\omega\}>$ but by Proposition 1.7 this is sufficient.)

(3) Existence: Let $<d_i:i\in(N^+)^*,d_\omega>$ be a sequence of (1) indiscernibles. Let $c_i=d_\omega-d_i$ for $i\in(N^+)^*$. Note that $c_1<c_2<c_3\cdots$. The c_i's are indiscernibles because the d_i's are indiscernibles. We have $2c_1=2(d_\omega-d_1)>(d_\omega-d_2)=c_2$. Hence $2c_1>c_2$. Also $(c_3-c_1)=(d_\omega-d_3)-(d_\omega-d_1)=d_1-d_3<2(d_1-d_2)=2(c_2-c_1)$. So the c_i's are case (3) indiscernibles. So we get exactly 3 types. □

<u>Corollary 2.2</u>: There are exactly 6 types of indiscernibles. They are (1), (2), (3) as in the theorem and their negatives.

<u>Proof</u>: If $\{c_i\}$ are indiscernibles and $c_1>0$ then by the theorem they are (1), (2) or (3). If $c_1<0$ then $\{-c_i\}$'s are indiscernibles with $-c_i>0$, so the $\{-c_i\}$'s are (1), (2) or (3). This determines the type of the c_i's, so we have at most 3 types with $c_1<0$. For existence of the negative cases, just let

$c_i=-d_i$ where the d_i's are (1), (2) or (3). □

B. Interdefinability

Even though there are 6 types of indiscernibles, from an algebraic point of view we only need to look at one of them. This is because they are "interdefinable" in the following sense:

Proposition 2.3: a) If a model A contains (1) indiscernibles of index set I where I has last element ω, then it contains (2) indiscernibles of index set $<\{0\}, I\setminus\{\omega\}>$. (b) Conversely, if A contains (2) indiscernibles of index set I, where I has first element 0, then it contains (1) indiscernibles of index set $<I\setminus\{0\},\{\omega\}>$.

Proof: (a) Let $<c_i>$ be (1) indiscernibles as in the hypothesis and let $d_i=c_i+c_\omega$ for $i\in I\setminus\{\omega\}$. By the existence proof of Theorem 2.1, the d_i's are (2) indiscernibles. We show that $<c_\omega,d_i:i\in I\setminus\{\omega\}>$ are (2) indiscernibles. Let $d_o=c_\omega$, where $0<I$. Pick any term $t(x_1,\ldots x_n)$. Then for any $\vec{i}>1$,

$$t(d_o,\vec{d}_i)>0 \leftrightarrow t'(d_i-\vec{d}_o,d_o)>0 \leftrightarrow t'(\vec{c}_i,c_\omega)>0$$

and $t(d_1,\vec{d}_i)>0 \leftrightarrow t'(d_i-\vec{d}_1,d_1)>0 \leftrightarrow t'(c_i-\vec{c}_1,c_\omega+c_1)>0$

Now, $<c_i-c_1:i\in\vec{i},c_\omega+c_1>$ are (1) indiscernibles and hence have the same type as $<c_i : i\in\vec{i}, c_\omega>$. So $t'(\vec{c}_i, c_\omega) \leftrightarrow t'(c_i - \vec{c}_1, c_\omega + c_1) > 0$ and hence $t(d_o, \vec{d}_i) \leftrightarrow t(d_1, \vec{d}_i)$. Similarly $t(d_o, \vec{d}_i) > 0 \leftrightarrow t(d_k, \vec{d}_i) > 0$ wherever $\vec{i} > k$. So $<d_o, d_i : i\in I\setminus\{\omega\}>$ are (2)-indiscernibles.

(b) Simply note that by (2) uniqueness of Theorem 2.1, the $<c_i - c_o, c_o>$ are (1)-indiscernibles. □

Thus for appropriate index sets I, (i.e., with a last element) the (2)-indiscernibles are recoverable from the (1)-indiscernibles and conversely. That is, type (1) I $\rightarrow$ type (2) $<\{0\}, I\setminus\{\omega\}>$ $\rightarrow$ type (1) $<(\{0\}, I\setminus\{\omega\})\setminus\{0\}, \{\omega\}> \cong$ type (1) I.

Similarly for type (2) to type (1) to type (2), when I has a first element.

<u>Remark 2.4</u>: A similar argument shows that the (1)-indiscernibles are also interdefinable with the (3)-indiscernibles.

<u>Proposition 2.5</u>: There exists a unique maximal set of (1)-indiscernibles in any model A up to order isomorphism of the index set.

<u>Proof</u>: By Zorn's Lemma there exists maximal sets. Suppose $<c_i : i\in I>$ and $<d_j : j\in J>$ are both maximal sets of (1)-indiscernibles. We will define an order isomorphism for the index sets. Pick any $i\in I$. Suppose that for each $j\in J$, $\forall n(nd_j < c_i)$ or

$\forall n(nc_i<d_j)$. Then $<d_j:j\in J>\cup\{c_i\}$ is a larger set of (1)-indiscernibles, contradicting maximality. Hence for each $i\in I$ there exists a $j\in J$ such that $\varepsilon d_j<c_i<Nd_j$ for some ε and N. Let $f(i)=j$. It is well defined by the (1)-indiscernibility of the d_j's. It is 1 to 1 by the indiscernibility of the c_i's. Suppose f was not onto. Say d_k was missed. Then for each c_i and for all $n, nc_i<d_k$ or $nd_k<c_i$. Hence $<c_i:i\in I>\cup\{d_k\}$ is a larger set of indiscernibles, and that contradicts maximally. Clearly f is order preserving and hence f is the desired order isomorphism. □

If the maximal (1)-indiscernible set is indexed by an I which has a greatest element then we can show there is a unique maximal index set for (2)-indiscernibles. However, things may not work out so nicely if I has no maximal element.

Proposition 2.6: If A has a maximal set of (1)-indiscernibles $<c_i:i\in N>$ then A has no infinite set of (2)-indiscernibles.

Proof: Suppose to the contrary that A contains such a set $<d_j:j\in J>$ with J infinite. Suppose that J has a subsequence indexed by N. Then by Proposition 2.3, A contains a set of (1)-indiscernibles with index set $< N\setminus\{1\},\{\omega\}>$. But this index set does not embed into N, contradicting the maximality of the c_i's. So J must have a subsequence indexed by N^*. Say

$d_1 > d_2 > d_3 \ldots$. Now by the maximality of I, $d_1 \approx c_i$ for some $i \in N$. Choose any $n > i$. Then $d_{n-1}-d_n, d_{n-2}-d_n, \ldots d_2-d_n, d_1-d_n, d_1$ is a sequence of (1)-indiscernibles. (From the definition of (2)-indiscernibles). But this gives us a chain of length $>i$ below c_i, contradicting maximality. So there could not exist such an infinite index set J. □

Thus there is not necessarily a unique maximal index set for (2)-indiscernibles. So although we have strong interdefinability conditions, it is easier to just work with the (1)-indiscernibles. And since the (1)-indiscernibles yield all the information about the (2) and (3)-indiscernibles we might as well.

C. Structure Theory

The maximal (1)-indiscernible index set I is a natural algebraic invariant. In this section we relate it to the standard terminology used in describing DOAG's (see Fuchs).

Definition: A system $[\Pi, B_\pi (\pi \in \Pi)]$ is called the skeleton of the DOAG A if

1) Π indexes the principal convex subgroups Σ_o of A in inverse order, and

2) If $D_\pi < C_\pi$ is a jump in Σ_o then $B_\pi = C_\pi / D_\pi$.

<u>Proposition 2.7</u>: If a maximal set of (1)-indiscernibles of A are $<c_i : i\in I>$ and A has skeleton $[\Pi, B_\pi(\pi\in\Pi)]$ then

a) $(I,<) \cong (\Pi,>)$

and b) $C_\Pi = \{x: \quad r\in Q\,(-rc_\pi<x<rc_\pi)\}$ and

$D_\Pi = \{x:\forall r\in Q\,(-rc_\pi<x<rc_\pi)\}$

<u>Proof</u>: Clear from Fuchs.

Note: Fuchs actually gives definitions which include the non-abelian case, so there are more details needed there.

<u>Remark 2.8</u>: The B_π's are subgroups of $(R,+)$. But the B_π's are not specified up to ordered group with unit isomorphism. In particular, the "irrationals" in B_π are dependent on which c_π is chosen from A.

We now look at which groups have maximal (1)-indiscernibles of index set I.

<u>Definition</u>: Let $W(A_i:i\in I) = \{x\in \prod_{i\in I} A_i$: such that the nonzero coordinates are a well founded sequence of $(I,>)\}$ = "Lexicographic Sum".

Note: Fuchs has things ordered in the opposite direction because Π is opposite to I. If we let the A_i's be ordered divisible abelian groups, then we can define + on

$W(A_i : i \in I)$ in the natural way and again get a divisible ordered abelian group.

By the Hahn embedding theorem, every DOAG A with skeleton $(B_i : i \in I)$ is embeddable into $W(B_i : i \in I)$. Hence the largest group with maximal (1)-indiscernible index set I is $W(R_i : i \in I)$ where $R_i = R$ (since $B_i \subseteq R$). The smallest group with maximal index set I is the "E-M model over I" namely $\oplus \sum_{i \in I} Q_i$. So the groups with maximal (1)-indiscernible index set I are extensions of $\oplus \sum_{i \; I \in i} Q$ by 1) replacing some of the Q_i's by $A_i \subseteq R$ where A_i is a DOAG and 2) allowing well founded sequences of c_i's (i.e., extending $\oplus \sum$ inside W(I)). Note that for extensions of kind 2) we require the new group be divisible, so we have to close under both group operations and divisibility. Also note that if I is finite then A is a divisible subgroup of $\oplus \sum_{i \in n} R_i$. Thus I really is a nice algebraic invariant of a model A.

D. Automorphisms

We now take a brief look at the automorphisms of certain DOAG's.

Proposition 2.9: Let $A = \oplus \sum_{i \in I} Q_i$ = the divisible group generated from the (1)-indiscernibles $\langle c_i : i \in I \rangle$. Let F be an automorphism on A. Then $F = F_3 \circ F_2 \circ F_1$ where

F_1 is the automorphism induced by an automorphism on $(I,<)$ and

F_2 is multiplication of c_i by $a_i>0$ and

F_3 is an isomorphism which takes $c_i \to c_i+p_i$ where p_i is a term containing only c_j's with $j<i$.

Proof: Suppose $F:c_i \to a_i c_{j_i}+p_i$ where $p_i \ll c_{j_i}$. (We will use $a \ll b$ to mean for all $n\in N$ $na<b$.) Then $i \to j_i$ must be an automorphism on $(I,<)$. (Since F is order preserving and and the c_i's are a maximal 1-indiscernible set.)

Let F_1 be the induced extension on A,

i.e., $q_{i_1}c_{i_1}+\ldots q_{i_n}c_{i_n} \to q_{i_1}c_{F(q_1)}+q_{i_2}c_{F(i_2)}+\ldots q_{i_n}c_{F(i_n)}$.

F_1 is clearly an automorphism. Now $F_1^{-1}\circ F:c_i \to a_i c_i+q_i$ where $q_i \ll c_i$. Let $F_2:c_i \to a_i c_i$ and extend it in the obvious way to all of A. Since $F_1^{-1}\circ F$ is an automorphism $a_i c_i+q_i>0$ and hence $a_i>0$. Thus F_2 is the desired kind of automorphism. Now $F_2^{-1}\circ F_1^{-1}\circ F: c_i \to c_i+\frac{1}{a_i}q_i$. Let $F_3=F_2^{-1}\circ F_1^{-1}\circ F$ and we have our result. □

Remark 2.10: Every automorphism on $(I,<)$ induces an automorphism on A and every map $c_i \to a_i c_i, a_i>0$ induces an automorphism on A. But not every $c_i \to c_i+p_i$ with $p_i \ll c_i$ is an automorphism. It is a monomorphism, but it may fail to be onto. For example, let $I= N^*$ and let A be the group generated from $<c_i:i\in I>$. Define $F:c_i \to c_i+c_{i+1}$ and extend it in the natural way. Then F is not onto because $c_1 \notin F(A)$ (i.e., $G(\sum_{i=1} a_i c_i) = q(c_1, \ldots, c_n) + a_n c_{n+1} \neq c_1$). So the study of F's on $\oplus \sum_{i \in I} Q_i$ is a study of which "F_3's" are onto.

(Note: There are F's which are onto but have no fixed points. e.g., $F: c_{2n+1} \to c_{2n+1} + c_{2n+2} + c_{2n+3}$, $c_{2n+2} \to c_{2n+2} + c_{2n+3}$). So in studying automorphisms or DOAG's it is useful to look at the maximal index set I.

E. Indiscernibles over Sets

We now look at what happens when constants are added to the language.

Definition: A set $\langle c_i : i \in I \rangle$ of a model A of a theory T is a set of order indiscernibles over B if they are order indiscernibles in the theory $T_B = T \cup \mathrm{Diag}(B)$ (i.e., $A \models \Theta(\vec{c}, \vec{b}) \leftrightarrow \Theta(\vec{d}, \vec{b})$ for any two increasing sequences $\vec{c}$ and $\vec{d}$).

In the case of DOAG's, $C = \langle c_i : i \in I \rangle$ is a set of indiscernibles over B when for every $c_1 < \ldots < c_n$ and $d_1 < \ldots < d_n$ in C and $\hat{b} \in B$ $\sum n_i c_i + \hat{b} > 0$ iff $\sum n_i d_i + \hat{b} > 0$.

Remark 2.11: Since $(p/q)b$ is definable in T_B we may assume that B is a DOAG. So it has a maximal set of (1)-indiscernibles, say $\langle b_j : j \in J \rangle$.

To begin, note that if $\{c_i\}$ are indiscernible over B, then $\{c_i\}$ is a set of indiscernibles (i.e., indiscernible $\equiv$ indiscernible over ϕ). Here $\{c_i\}$ or their negatives must be of the form (1), (2) or (3) of Theorem 2.1. So we will divide the classification theorem into three lemmas.

Lemma (1): The types of (1)-indiscernibles over B are in 1 to 1 correspondence with the proper Dedekind cuts D of J. (i.e., $b_j \in D \leftrightarrow b_j < x \in \tau$).

<u>Proof</u>: Uniqueness: If $<c_i>$ is a set of indiscernibles then they must all lie in the same cut of the $\{b_i\}$'s. Furthermore, they must lie in a proper cut since there are an infinite number of them. Suppose C is any set of indiscernibles in the proper cut D. Then $b_j \in D \rightarrow b_j < c_1 \rightarrow b_j < c_1 < \frac{1}{n} c_2 \rightarrow nb_j < c_2 \rightarrow nb_j < c_1$. Similarly, $b_k \notin D \rightarrow c < b_k \rightarrow nc < b_k$. So $\{c_i\}_{i \in I} \cup \{b_j\}_{j \in J}$ are a set of (1)-indiscernibles (by the existence proof of Theorem 2.1). But we need to show more. We need to know that for any $\hat{b} \in B$ $\sum n_i c_i + \hat{b} \gtrless 0$ is determined (independent of the particular choice of C). If all the n_i's are zero it is obviously determined, so assume $n_k \neq 0$. We show that in this case the inequality is also determined. Well $\hat{b} \approx r\, b_i$ for some r and b_j by the maximality of the $\{b_i\}$'s. That is $q_1 b_i < \hat{b} < q_2 b_i$ wherever $q_1 < r < q_2$. (Note that r may be irrational.) If $b_i > \{c_i\}$'s (i.e.,

$b_1 \notin D$) then the sign of r determines the inequality. If $b_i < \{c_i\}$'s, then $\sum n_i c_i \gtrless 0$ determines the inequality. Since the c_i's are of type (i) this inequality is determined. So in any case all the formulas $\sum n_i x_i + \hat{b} \gtrless 0$ are determined. This type of formulas is realized by any C in the cut of D, so we have uniqueness.

Existence: Use the compactness theorem to find (1)-indiscernibles $\langle c_i \rangle$ in the D cut of the b_j's. The above procedure for determining inequalities works and is in fact independent of the indices of the c_i's. Hence the c_i's are indiscernible over B.

This establishes the 1 to 1 correspondence. □

Before beginning the next lemma we will describe the 1-types over B. Pick any DOAG $A \supseteq B$ and any $\hat{x} \in A$, $\hat{x} > 0$. Let $G_{\hat{x}}$ be the collection of atomic formulas over B realized by $\hat{x}$. By QE $G_{\hat{x}}$ determines a 1-type. Since we are working with indiscernibles we are only interested in those G_x's which contain $x \neq b$ for every $b \in B$ (i.e., the nonprinciple types). Now any G_x gives an approximation to x by b's in B. For example, G_x may say that x is approximately b with an error of less than b_j. So each $x - b > 0 \in G_x$ corresponds to a cut in the b_j's indicating the "error term". Let D_x = "inf of the cuts". That is, define D_x by: $\forall j \in J (j > D_x \leftrightarrow \exists \hat{b} \in B (0 < x - \hat{b} << b_j))$ and

$j=D_x \leftrightarrow \exists \hat{b}\in B \ \ \exists q_1,q_2\in Q(0<q_1b_j<x-\hat{b}<q_2b_j) \wedge \forall k \leq j(\sim k>D_x)$.

$j<D_x \leftrightarrow \sim(j>D_x) \wedge \sim(j=D_x)$.

We will say $b>D_x$ iff $b\hat{\sim}rb_j$ and $j>D_x$. Similarly for $b=D_x$ and $b<D_x$.

Lemma (2): The types τ of (2)-indiscernibles over B may be classified as follows. First τ contains some non-principle type G_x. For those G_x such that $b>D_x$ with x_1-b in a proper cut D_x, the extra condition

a) $0<x_1-b<x_2-x_1$ is in τ

will completely specify τ. For any G_x and for any proper cut D' of $\{b_j\text{'s}\}$ with $D' \leq D_x$ the extra condition

$b)_{D'}$ $\sim(0<x_1-b<x_2-x_1)$ is in τ for each

$b\in B$, and $b_j<x_2-x_1<b_k$ is in τ for each $b_j<D'<b_k$

will completely specify τ. These are the only possibilities.

Proof: First note that if $<c_i>$'s are (2)-indiscernibles, then $d_i=c_i-c_1$ are (1)-indiscernibles and so must lie in some fixed cut D' of the $\{b_i\}$'s. Suppose $0<c_1-b<b_k<c_2-c_1$. Then $c_2-c_1<c_2-b$ by adding c_2-c_1 to both sides of $0<c_1-b$. But then $c_2-b<b_k<c_2-c_1$ by indiscernibility, a contradiction. Now suppose that for some fixed b, just $0<c_1-b<c_2-c_1$. Then by the above the d_i's and c_1-b must lie in the same cut of the $\{b_j\}$'s. Indeed, $c_1-b'<b_i<n(c_1-b)<c_2-c_1$ also yields a

contradiction so c_1-b and c_i-c_1 are in the cut D_x. Note that this forces D_x to be a proper cut in the $\{b_j\}$'s (i.e., $D_x \neq j$ for any $j \in J$). Suppose also, that $b \leq D_x$. Then $b < D_x$, since D_x is a proper cut. Now $2(c_1-b) < c_2-c_1$ by hypothesis and indiscernibility. So $2c_1-2b+c_1 < c_2$ and hence $2c_1 < c_2$, which is a contradiction to the fact that the c_i's are type (2). So we must have $b > D_x$. So if there exists c_1's of type (2) with a b s.t. $0 < c_1-b < c_2-c_1$ then we must satisfy the conditions about case (a). We now show that case (a) is well defined. Suppose for some G_x that both b,b* satisfy the assumption for case (a). Then c_1-b and c_1-b^* are in D_x. Since D_x is a proper cut we have $|b-b^*| < D_x$. Also c_2-c_1 is in D_x, by above. So $c_1-b < c_2-c_1 \rightarrow n(c_1-b) < c_2-c_1 \rightarrow c_1-b+b-b^* < c_2-c_1 \rightarrow c_1-b^* < c_2-c_1$. (And conversely.) So $0 < x_1-b < x_2-x_1 \in \tau$ iff $0 < x_1-b^* < x_2-x_1 \in \tau$. On the other hand, if we are not in case (a) then for every $b \in B$ $c_1-b > 0 \rightarrow c_1-b > c_2-c_1$. Hence the c_i-c_1's are in a cut $D' \leq D_x$. Indeed D' is a proper cut because there are an infinite number of c_i-c_1's in D'. So hence we satisfy the conditions for case $(b)_{D'}$. Thus the outline is a partition of the types. It remains to show existence and uniqueness for each case.

Case (a): Uniqueness: Suppose C is a set of indiscernibles realizing G_x, satisfying the conditions for (a) with b, and such that $0<c_1-b<c_2-c_1$. By above, c_1-b and c_i-c_1 are in the cut D_x. Now,

$$\sum m_i c_i + b' > 0 \leftrightarrow m(c_1-b)+mb+\sum m_i d_i + b' > 0.$$

$$\leftrightarrow m(c_1-b) + \sum m_i d_i + b'' > 0 .$$

If $b''>D_x$ then its sign determines the inequality. If $b''<D_x$ then the sign of $m(c_1-b) + \sum m_i d_i$ determines the inequality (unless $m=0=m_1=\ldots=m_k$ and then the sign of b'' does). Now $c_1-b<c_2-c_1 < \frac{1}{n}(c_3-c_1)$. Hence $n(c_1-b)<(c_2-c_1)$. So $\langle c_1-b, d_i\rangle$ is a sequence of (1)-indiscernibles, and hence the sign of $m(c_1-b)+\sum m_i d_i$ is determined. So these conditions determine a unique type.

(a) Existence: Suppose we have an appropriate G_x for (a). By Lemma (1) choose d_i's which are (1)-indiscernible over B and in the cut D_x. By assumption we may fix a b such that $b>D_x$ and $x-b$ is in the cut D_x. Let $c_i=b+d_i$. Then $c_i-b=d_i$ and $c_i-c_1=d_i-d_1>nd_1=n(c_1-b)$. Note that since the d_i's are indiscernible over B, the c_i's are indiscernible over ϕ. In fact they are (2)-indiscernible since $b>D_x$. Now we show c_1 is of type G_x. Well for $b'>0$,

$$x-b'>0\in G_x \rightarrow (x-b)+(b-b')>0\in G_x \rightarrow b>b' \quad \text{or} \quad b'-b<D_x.$$

While $c_1-b'>0 \rightarrow d_1+b-b'>0 \rightarrow b>b'$ or $b'-b<D_x$.

So c_1 does realize the type G_x. It remains to show that the c_i's are indiscernibles over B. Well $n(c_i-b)=nd_i<d_k-d_i=c_k-c_i$. Thus $<c_i-b,c_k-c_i:k>i>$ are (1)-indiscernibles for any i. Since they lie in D_x, we can apply the algorithm in Lemma (1) to decide any inequality. Since the algorithm is independent of the indices $\sum m_i c_{i_j}+b'>0 \leftrightarrow \sum m_i c_i+b'>0$. Thus the c_i's are indiscernibles over B.

Case (b): Uniqueness: Suppose we have a set of indiscernibles with $c_1-b>0 \rightarrow c_2-c_1<c_1-b$ and such that the c_i-c_1's lie in D'. Let $d_i=c_i-c_1$. For any b such that $c_1-b>0$ we have $nd_i=n(c_i-c_1)<c_1-b$. Thus if $c_1-b>0$, $<d_i:i>1, c_1-b>$ is a sequence of (1)-indiscernibles over ϕ. Now we test if $nc_1 + \sum m_i d_i-b'>0$. We may assume $n\neq 0$ because otherwise the inequality is determined. Now $nc_1+\sum m_1-d_i-b'>0 \rightarrow n(c_1-b'') + \sum m_i d_i>0$.

Case (i): $c_1-b''>0$ (i.e. $x_1-b''\in G_x$) then $(c_1-b'')>nd_i$ so $n(c_1-b'')+ \sum m_i d_i>0$ iff $n>0$.

Case (ii): $c_1-b''<0$. Then $c_2-b''<0$ and hence $c_2-c_1<b''-c_1$. So $nd_i=n(c_i-c_1)<b''-c_1$. Thus $n(c_1-b'')+\sum m_i d_i>0$ iff $n<0$.

So in either case the formula is determined and we have uniqueness.

Existence $(b)_{D'}$: Suppose we have G_x, D' such that D' is a proper cut $\leq D_x$. Let $\hat{x}$ realize G_x. Suppose $\hat{x}-b<0$. Then there is no $b_j, j\in J$ such that $b-\hat{x}<mb_j$ and $b_j<D_x$. Otherwise, $0<\hat{x}-b+mb_j<2mb_j \rightarrow D_x<b_j$, a contradiction. So we may choose (1)-indiscernibles $\langle d_i\rangle$ over B in D' satisfying $\hat{x}-b>0 \rightarrow \hat{x}-b>d_i$ and $b-\hat{x}>0 \rightarrow b-\hat{x}>d_i$ (by compactness and the fact that D' is a proper cut $<D_x$). Let $c_i=\hat{x}+d_i$. If $c_1-b>0$ then $\hat{x}+d_1-b>0$, hence $\hat{x}-b>-d_1$. Suppose $0<b-\hat{x}$. Then $b-\hat{x}>d_2$ by choice of the d_i's. But then $\hat{x}-b+b-\hat{x}>d_2-d_1$, a contradiction. So $c_1-b>0 \rightarrow \hat{x}-b>0$. Thus $\hat{x}-b>d_2$ by the choice of the d_i's. So $c_2-c_1 = d_2-d_1<d_1<\hat{x}-b<(\hat{x}-b)+d_1=c_1+b$ (i.e., $c_2-c_1<c_1-b$ whenever $c_1-b>0$). Also $c_2-c_1=d_2-d_1$ which is in the cut D'. We show the c_i's are indiscernibles over B. $\sum m_i c_{j_i}+b'>0 \leftrightarrow \sum m\hat{x}+\sum m_i d_{j_i}+b'>0 \leftrightarrow n\hat{x}+\sum m_i d_{j_i}+b'>0$ where $n=\sum m_i$. We may assume $n\neq 0$, since the d_i's are indiscernible over b'. So, $n\hat{x}+\sum m_1 d_{j_i}+b'>0 \leftrightarrow n(\hat{x}-b'')+\sum m_i d_{j_i}>0$. Now $\langle d_{j_1},\ldots d_{j_n},|x-b''|\rangle$ is a sequence of (1)-indiscernibles, so $n|\hat{x}-b''|+\sum m_i d_{j_i}>0 \leftrightarrow n(\hat{x}-b'')>0$. Hence the c_i's are indiscernibles over B. Note that since $\hat{x}>nd_i$ and $c_2-c_1=d_2-d_1$, they are in fact (2)-indiscernible over B. □

Lemma (3): The types τ of (3)-indiscernibles over B may be classified as follows: First τ contains some non-principle type G_x. For those G_x such that $b>D_x$ with $b-x_1$ in a proper cut D_x, the extra condition

a) $0<b-x_1<x_\omega-x_1$ is in τ

will completely specify τ. For any G_x and for any proper cut D' of $\{b_j's\}$ with $D'\leq D_x$ the extra condition

b)$_{D'}$ $\sim(0<b-x_1<x_\omega-x_1)$ is in τ for each

$b\in B$, and $b_j<x_\omega-x_1<b_k$ is in τ for each $b_j<D'<b_k$

will completely specify τ. These are the only possibilities.

Proof: Essentially the same as in Lemma (2). □

Theorem 2.11: The indiscernible types over B are classified by Lemmas (1), (2) and (3) and their negatives.

Proof: By above. □

Remark 2.12: One can now derive as a Corollary the indiscernibles for the theory of divisible ordered abelian groups with unit.

3. REAL CLOSED FIELDS

The main objective in this chapter will be the classification of the order indiscernibles for RCF's. We will also describe how the types relate to each other and how they describe models.

A. Classification of the Indiscernibility Types

Since the classification is rather long we will break up the theorem into several lemmas. We will be able to obtain most of the classifications by just looking at those order indiscernibles $<c_i>$ for which $c_1 > n$ for every $n \in N$. This condition will sometimes be abbreviated as c_1 is infinite, or as $c_1 > Q$, or as $c_1 > R$.

<u>Lemma I</u>: There is exactly one type of indiscernible τ containing $x_1 > Q$ and $x_1^{1+\varepsilon} < x_2$ for some $\varepsilon \in Q^+$.

<u>Proof</u>: Uniqueness: In the restricted language of $1, \cdot, <$, the positive elements of a RCF form a DOAG. So in particular, any indiscernible type restricted to this language must be a DOAG indiscernible type. If we add Q^+ to the language then the type is a DOAG type over Q^+. Now since $x_1^{1+\varepsilon} < x_2 \in \tau$ it must be of type (i). In fact, since $x_1 > Q \in \tau$ the DOAG type over Q is

completely determined. In particular all questions of the form $q_1x_1^{n_1}x_2^{n_2}\ldots x_k^{n_k}\gtrless 1\epsilon\tau$ are determined. Namely, $q_1x_1^{n_1}\ldots x_k^{n_k}>1\epsilon\tau$ iff the highest indexed exponent which is not zero is positive, or all the n_i's are 0 and $q_1>1$. Of course we need to decide all polynomial questions to get uniqueness as a RCF indiscernible type. Well for any polynomial one of the monomials is largest in absolute value, by above. Suppose $p(x_1,\ldots x_n)=\sum m_i$ where m_1 is the largest monomial in absolute value. Then for every i $|m_1| \geq x_{i_j}^{\varepsilon}|m_i|$ for some x_{i_j} (for instance the highest indexed variable in m_1/m_i). So in particular, $|m_1| \geq \sum|m_i|$. Hence $m_1 + \sum_{i>1} m_i>0$ iff $m_1>0$ iff the leading coefficient of m_1 is positive. So all polynomial inequalities are determined and we have uniqueness.

Existence: By direct construction or by compactness we can find c_i's such that $c_1>n$ for all n N and such that $c_k^n<c_{k+1}$ for all $n,k\epsilon$ N. Now $p(c_1,c_2,\ldots c_n)>0$ iff the procedure given in the uniqueness says it is. This is because the procedure's correctness only depends on the conditions that we already have. This procedure is also independent of the particular choice of indices, so $p(c_1,c_2\ldots c_n)>0$ iff $p(c_{i_1},\ldots c_{i_n})>0$ where $i_1<i_2\ldots i_n$. That is, the c_i's are RCF indiscernibles. □

We will usually call this type of indiscernible, I-indiscernibles, or perhaps $+\infty$ indiscernibles. The reason for the "$+\infty$" will be seen shortly.

Now suppose we have c_i's spaced far apart but not quite as far as in Lemma I. We then have a situation much like that for DOAG's.

Lemma II: There are exactly two types of indiscernibles with $x_1 > Q$ and such that $2x_1 < x_2 < x_1^2$ depending on

A. $\left(\frac{x_3}{x_1}\right) > \left(\frac{x_2}{x_1}\right)^2$

or

B. $\left(\frac{x_3}{x_1}\right) < \left(\frac{x_2}{x_1}\right)^2$

Proof: First note that $x_2 < x_1^2$ together with formulas A or B is just the multiplicative version of DOAG types (2) or type (3), respectively. Indeed the proofs will use the same coding scheme. Let $\langle c_i \rangle$ be any sequence of indiscernibles for Lemma II. By Lemma I, we have $c_2 < c_1^{1+\varepsilon}$ for all $\varepsilon \in Q^+$. We also have $Hc_1 < c_2$ for all $H \in Q^+$. For if $c_2 \leq Hc_1$ for some $H \in Q^+$ then $c_3 < Hc_2$ and so $c_2 < H^2 c_1$. Repeating this argument we get $c_2 > Hc_1^n$ for all $n \in N$. In particular, since $\langle c_i \rangle$ satisfy $x_2 > 2x_1$ we have the desired result. Now clearly we are in Case A or in Case B, so it remains to show uniqueness and existence for each type.

Case IIA: Uniqueness: Suppose $\langle c_i \rangle$ satisfy conditions IIA. Let $d = \frac{c_n}{c_1}$ for $n>1$. By Theorem 2.1, the d_i's are DOAG type (i) indiscernibles, so $d_2^2<d_3$. Also $\frac{c_2}{c_1}>Q$, so the d_i's $>Q$. Again by Theorem 2.1 we have $d_i^n<c_1$. Hence by Lemma I, $\langle d_i\colon i>1, c_1\rangle$ is a sequence of I-indiscernibles. Now any $p(\vec{c}_i)>0 \leftrightarrow q(\vec{d}_i, c_1)>0$ for the natural q. So the type is determined independent of the particular choice of c_i's. Hence we have uniqueness.

Existence: Let $\langle d_i\colon i\in N^+, d_\omega\rangle$ be a sequence of I-indiscernibles. Let $c_i=d_i d_\omega$ for $i\in N^+$. Since the d_i's are indiscernibles so are the c_i's. By Theorem 2.1, $c_2<c_1^2$ and $(\frac{c_3}{c_1})>(\frac{c_2}{c_1})^2$. Also, $2c_1=d_1d_\omega<d_2d_\omega=c_2$ and $Q<d_1d_\omega=c_1$. So the c_i's satisfy the formulas of IIA.

Case IIB: Uniqueness: Let $\langle c_i\colon i\in N^+, c_\omega\rangle$ be IIB indiscernibles. Let $d_i=\frac{c_\omega}{c_i}$. By Theorem 2.1, $\langle d_i\colon i\in (N^+)^*, c_\omega\rangle$ are DOAG (1)-indiscernibles. Also $\frac{c_\omega}{c_i}>Q$ so by Lemma I, it is a sequence of RCF I-indiscernibles. The c_i's are definable from these, and so the type is determined.

Existence: Let $\langle d_i\colon i\in (N^+)^*, d_\omega\rangle$ be a sequence of I-indiscernibles. Let $c_i=\frac{d_\omega}{d_i}$. Since the d_i's are indiscernible so are the c_i's. We have $c_1{}^2>c_2$ and $(\frac{c_3}{c_1})<(\frac{c_2}{c_1})^2$ by Theorem 2.1. Also $Q<\frac{d_\omega}{d_i}=c_i$ and $2c_1=2\frac{d_\omega}{d_1}<\frac{d_\omega}{d_2}=c_2$. Hence the c_i's are IIB-indiscernibles. $\square$

Now in the remaining cases we have $x_1<x_2<(1+\varepsilon)x_1$. So we need a way to classify the types when x_2 and x_1 are closer together. We will use the r such that $x_1^{q_1}<x_2-x_1<x_1^{q_2}$ for $q_1<r<q_2$ (i.e. $x_2-x_1 \approx x_1^r$). Lemma I corresponds to the case $r=+\infty$. Lemma II A,B are cases where r=1. The remaining cases have $r\leq 1$. For r rational there are two possibilities, $x_2-x_1<x_1^r$ and $x_1^r<x_2-x_1$. This will lead us to Lemma III and Lemma IV, respectively. Lemma V will be the case where r is irrational. The only other possibility is $r=-\infty$, and this will by Lemma VI. Thus Lemmas I-VI will be a complete classification of the types with $x_1>Q$.

<u>Lemma III</u>: For each rational $r\leq 1$ there are exactly six types of indiscernibles containing $x_1>Q$ and $x_1^q<x_2-x_1$ for $q<r$ and $x_2-x_1<x_1^r$. They are classified by:

A. $(x_3-x_1)>2(x_2-x_1)$

1. $\dfrac{x_1^r}{x_i-x_1}$ are type I (i.e., the $\dfrac{x_1^r}{x_i-x_1}$'s satisfy the type I formulas).

2. $\dfrac{x_1^r}{x_i-x_1}$ are type IIA.

3. $\dfrac{x_1^r}{x_i-x_1}$ are type IIB.

B. $(x_3-x_1)<2(x_2-x_1)$

1. $\dfrac{x_\omega^r}{x_\omega-x_i}$ are type I

2. $\dfrac{x_\omega^r}{x_\omega-x_i}$ are type IIA.

3. $\dfrac{x_\omega^r}{x_\omega-x_i}$ are type IIB.

Proof: Note that we are appealing to Proposition 1.7 in describing the B cases. Let $<c_i>$ be any sequence of indiscernibles satisfying II(r) (i.e. satisfy the initial hypothesis of Lemma II for r). First we show $c_2-c_1>hc_1$ for all $h\epsilon Q^+$. For suppose $c_2-c_1>hc_1^r$ for some $h\epsilon Q^+$. Then $c_3-c_2>hc_2^r$ by indiscernibility. By adding these two inequalities we get $hc_1^r+hc_2^r<c_3-c_1$. Hence $2hc_1^r<c_3-c_1$. And thus $(2h)c_1^r<c_2-c_1$. Iterating this procedure we get $Hc_1^r<c_2-c_1$ for all $H\epsilon Q^+$, which contradicts the fact that the c_i's satisfy $x_2-x_1<x_1^r$. Now let us see that each possible type is in exactly one of the six cases (i.e., we want to check that the lemma actually partitions the cases). Well clearly any type is either A or B but not both. Suppose $<c_i>$ realizes III(r)A. Then since $c_2-c_1<hc_1^r$ for all $h\epsilon Q^+$ we have $H<\dfrac{c_1}{c_i-c_1}$ for all $H\epsilon Q^+$ (i.e., $Q<\dfrac{c_1^r}{c_i-c_1}$). Also since $(c_3-c_1)>2(c_2-c_1)$ we have $\dfrac{1}{c_2-c_1}>2\dfrac{1}{c_3-c_1}$, so $\dfrac{c_1^r}{c_2-c_1}>2\dfrac{c_1^r}{c_3-c_1}$. Hence

$\frac{c_1^r}{c_i - c_1} > 2\,\frac{c_1^r}{c_j - c_1}$ for $i<j$. This means $\langle \frac{c_1^r}{c_i - c_1} : i \in (N^+)^* \setminus \{1\} \rangle$ are Lemma I or Lemma II indiscernibles. So any IIIA type must be in case A1, A2 or A3. Now suppose $<c_i : i \in N^+, c_\omega>$ are III(r)B-indiscernibles. Then $Q < \frac{c_i}{c_\omega - c_i}$ by above and indiscernibility. Now $(c_3 - c_1) < 2(c_2 - c_1) \rightarrow (c_3 - c_1) < (1+h)(c_2 - c_1)$ for all $h \in Q^+$ by an indiscernibility argument. Thus $(c_3 - c_2) < h(c_2 - c_1)$ for all $h \in Q^+$. So $\frac{H}{c_3 - c_1} < \frac{1}{c_3 - c_2}$ for all $H \in Q^+$. Thus $\frac{2}{c_\omega - c_i} < \frac{1}{c_\omega - c_j}$ for $i<j$ by indiscernibility. Hence $\frac{2c_\omega^r}{c_\omega - c_i} < \frac{c_\omega^r}{c_\omega - c_j}$ for $i<j$. This means $\langle \frac{c_\omega^r}{c_\omega - c_i} : i \in N^+ \rangle$ are Lemma I or Lemma II indiscernibles. So any III(r)B type must be in case B1, B2, or B3. It remains to show that we get exactly one type for each of the cases.

Cases A1, A2, A3: Uniqueness: Let $<c_i : i \in N^+>$ be any sequence of indiscernibles in A1, A2 or A3. By the case hypothesis $c_1^{r-\varepsilon} < c_i - c_1$ and so $\frac{c_1^r}{c_i - c_1} < c_1^\varepsilon$. Hence $\left(\frac{c_1^r}{c_i - c_1}\right)^n < c_1 \ \forall n$.

Case A1: $\frac{c_1^r}{c_i - c_1}$ are I-indiscernibles, hence $\langle \frac{c_1^r}{c_i - c_1} : i \ (N^+)^* \setminus \{1\}, c_1 \rangle$ are I-indiscernibles.

Case A2: $\frac{c_1^r}{c_i - c_1}$ are IIA indiscernibles, so

$$\left\langle \frac{\frac{c_1^r}{c_i - c_1}}{\frac{c_1^r}{c_\omega - c_1}}, \frac{c_1^r}{c_\omega - c_1} \right\rangle$$

are Indiscernibles, by the proof of Lemma IIA.

Hence $\left\langle \frac{c_\omega - c_1}{c_i - c_1}, \frac{c_1^r}{c_\omega - c_1}, c_1 \right\rangle$ are I-indiscernibles

Case A3: $\frac{c_1^r}{c_i - c_1}$ are IIB, so

$$\left\langle \frac{\frac{c_1^r}{c_2 - c_1}}{\frac{c_1^r}{c_i - c_1}}, \frac{c_1^r}{c_2 - c_1} \right\rangle$$ are I-indiscernibles, by

the proof of Lemma IIB.

Hence $\left\langle \frac{c_i - c_1}{c_2 - c_1}, \frac{c_1^r}{c_2 - c_1}, c_1 \right\rangle$ are I-indiscernibles.

Now in each of the cases A1, A2, and A3 the c_i's may be encoded as I-indiscernibles. Thus III(r)A1, III(r)A2, and III(r)A3 specify a unique type.

Existence: Let $\langle d_i : i \in (N^+)^* \rangle$ be I, IIA or IIB indiscernibles and choose $e > d_i^n$ $\forall n \in N$ (by compactness). Let $c_i = \frac{1}{d_i} e^r + e$ (i.e., undo the coding given in the uniqueness part).
We check that the c_i's are III(r)-indiscernibles of case A1, A2 or A3 depending on the choice of d_i's as I, IIA or IIB.
First it is easy to verify that the d_i's are in fact "indiscernible over e_ω", that is, $p(\vec{d}_{i_j}, e_\omega)$ 0 $\leftrightarrow$ $p(\vec{d}_i, e_\omega) > 0$ whenever

$d_{i_1} > d_{i_2} \ldots > d_{i_n}$. Hence the c_i's are indiscernibles, $(c_2-c_1) = (\frac{1}{d_2} - \frac{1}{d_1})e^r = \frac{d_1-d_2}{d_1d_2} < e^r < c_1^r$. So $c_2-c_1 < c_1^r$. Also, $\frac{d_1d_2}{d_1-d_2} < d_3 < e_1^{\varepsilon/2}$ so, $c_2-c_1 = \frac{d_1-d_2}{d_1d_2} e^r > e^r e^{-\varepsilon/2} = e^{r-\varepsilon/2} > c_1^{r-\varepsilon}$ (i.e., $c_2-c_1 > c_1^q$ for all $q<r$), thus the c_i's are III(r). We have $c_2-c_1 = \frac{d_1-d_2}{d_1d_2} e^r$ and $c_3-c_1 = \frac{d_1-d_3}{d_1d_3} e^r$. Now we check that $(c_3-c_1) > 2(c_2-c_1)$. Well,

$$(c_3-c_1) > 2(c_2-c_1) \leftrightarrow \frac{d_1-d_3}{d_1d_3} e^r > 2\frac{d_1-d_2}{d_1d_2} e^r$$

$$\leftrightarrow (d_1-d_3)d_1d_2 > 2(d_1-d_2)d_1d_3$$

$$\leftrightarrow d_2 > 2\left(\frac{d_1-d_3}{d_1-d_2}\right) d_3$$

Now $1 < \frac{d_1-d_3}{d_1-d_2} < 2$ by the choice of the d_i's. (Remember the d_i's are ordered backwards.) Hence,

$$d_2 > 2\left(\frac{d_1-d_3}{d_1-d_2}\right) d_3 \leftrightarrow d_2 > 2d_3 \leftrightarrow \text{Truth.}$$

So the c_i's realize a III(r)A type. Now

$$\frac{c_1^r}{c_i-c_1} = \frac{\left(\frac{1}{d_1} e^r + e\right)^r}{\left(\frac{1}{d_i} - \frac{1}{d_1}\right)e^r} = \frac{d_1d_i}{d_1-d_i} \quad \frac{\left(\frac{1}{d_1} e^r + e\right)^r}{e^r} \quad .$$

$$1 < \frac{d_1}{d_1 - d_i} < 1+\varepsilon \text{ and } 1 < \left(\frac{\frac{1}{d_1} e^r + e}{e^r}\right)^r < 1+\varepsilon \quad \text{for all } \varepsilon \epsilon Q^+ .$$

So $d_i < \frac{c_1^r}{c_i - c_1} < (1+\varepsilon)d_i$. In particular, the $\frac{c_1^r}{c_i - c_1}$ are type I, IIA or IIB indiscernibles depending on the choice of the d_i's. (This is easy to verify.) So the c_i's are III(r)A1, III(r)A2 or III(r)A3 depending on the choice of the d_i's. Hence, we have existence for all the A cases.

Cases B1, B2, B3: Uniqueness: Let $\langle c_i : i \in N^+, c_\omega \rangle$ be type III(r)B indiscernibles by the case hypothesis $\left(\frac{c_1^r}{c_i - c_1}\right)^n < c_1$ for all n. Hence,

$\left(\frac{c_1}{c_\omega}\right)^{rn} \left(\frac{c_\omega^r}{c_\omega - c_1}\right)^n < c_1$ for all n. Now $1 < \frac{c_1}{c_\omega} < 1+\varepsilon$, so that

$$\left(\frac{c_\omega^r}{c_\omega - c_1}\right)^n < c_1 < c_\omega \quad \text{for all n.}$$

Case B1: $\frac{c_\omega^r}{c_\omega - c_i}$ are I-indiscernibles, hence $\left\langle \frac{c_\omega^r}{c_\omega - c_i} , c_\omega \right\rangle$ are I-indiscernibles.

Case B2: $\frac{c_\omega^r}{c_\omega - c_i}$ are IIA. Then

$$\left\langle \frac{\frac{c_\omega^r}{c_\omega - c_i}}{\frac{c_\omega^r}{c_\omega - c_1}} , \frac{c_\omega^r}{c_\omega - c_1} \right\rangle \quad \text{are I-indiscernibles}$$

so $\left\langle \frac{c_\omega - c_1}{c_\omega - c_i}, \frac{c_\omega^r}{c_\omega - c_1}, c_1 \right\rangle$ are I-Indiscernibles.

Case B3: $<c_i: i\in N^+, c_\omega, c_{\omega+1}>$ are III(r)B3. Then

$\frac{c_{\omega+1}^r}{c_{\omega+1} - c_i}$ are II B. Thus $\left\langle \frac{\frac{c_{\omega+1}^r}{c_{\omega+1}-c_\omega}}{\frac{c_{\omega+1}^r}{c_{\omega+1}-c_i}}, \frac{c_{\omega+1}^r}{c_{\omega+1}-c_\omega} \right\rangle$ are

I-indiscernibles. So $\left\langle \frac{c_{\omega+1}-c_i}{c_{\omega+1}-c_\omega}, \frac{c_{\omega+1}^r}{c_{\omega+1}-c_\omega}, c_{\omega+1} \right\rangle$

are I-indiscernibles.

In each case the c_i's may be encoded as I-indiscernibles and so III(r) B1, III(r) B2, and III(r) B3 each uniquely specify a type.

Existence: Let $<d_i: i\in N^+, d_\omega>$ be type I, IIA, or IIB indiscernibles and choose $e > d_\omega^n$ for all n.
Let $c_i = e - \frac{1}{d_i} e^r$ for $i\in N^+$. As in the A cases the c_i's are indiscernibles. $c_2 - c_1 = (\frac{1}{d_1} - \frac{1}{d_2}) e^r = \frac{d_2 - d_1}{d_1 d_2} e^r < c_1^r$. So $c_2 - c_1 < c_1^r$. $c_2 - c_1 = \frac{d_2 - d_1}{d_1 d_2} e^r > e^{r-\varepsilon} > c_1^{r-\varepsilon}$. Hence the c_i's

are III(r). We want $(c_3-c_1)<2(c_2-c_1)$. Well,

$$(c_3-c_1)<2(c_2-c_1) \leftrightarrow \frac{d_3-d_1}{d_3d_1} e^r < 2 \frac{d_2-d_1}{d_2d_1} e^r$$

$$\leftrightarrow \frac{d_3-d_1}{d_3} \cdot \frac{d_2}{d_2-d_1} < 2$$

Now $1-\varepsilon < \frac{d_i-d_1}{d_i} < 1$ for $i>1$, since the d_i's are I, IIA or IIB.

So $\frac{d_3-d_1}{d_3} \cdot \frac{d_2}{d_2-d_1} < 2$. By above, this means $(c_3-c_1)<2(c_2-c_1)$.
So we are in a case III(r)B. Also,

$$\frac{c_\omega^r}{c_\omega-c_i} = \frac{(e-\frac{1}{d_\omega} e^r)^r}{(\frac{1}{d_i}-\frac{1}{d_\omega})e^r} = \frac{d_\omega d_i}{d_\omega-d_i} \cdot \frac{(e-\frac{1}{d_\omega} e^r)^r}{e^r} .$$

$$1 < \frac{d_\omega}{d_\omega-d_i} < 1+\varepsilon \quad \text{and} \quad 1-\varepsilon < \frac{(e-\frac{1}{d_\omega} e^r)^r}{e^r} < 1 \quad \text{and so}$$

$(1-\varepsilon)d_i < \frac{c_\omega^r}{c_\omega-c_i} < (1+\varepsilon)d_i$. Hence the $\frac{c_\omega^r}{c_\omega-c_i}$ are I,

IIA or IIB depending on the choice of the d_i's. So we have existence for the B cases. □

It is interesting to see the interconnection between the DOAG indiscernible types and those for RCF. For each choice of c_i's, let the d_i's be a natural encoding of the c_i's. Then we have:

	$d_1^2<d_2$	$d_1^2>d_2$ $\left(\frac{d_3}{d_1}\right) > \left(\frac{d_2}{d_1}\right)^2$	$d_1^2>d_2$ $\left(\frac{d_3}{d_1}\right) < \left(\frac{d_2}{d_1}\right)^2$
$2c_1<c_2$	I	IIA	IIB
$2c_1>c_2$ $(c_3-c_1)>2(c_2-c_1)$	IIIA1	IIIA2	IIIA3
$2c_1>c_2$ $(c_3-c_1)<2(c_2-c_1)$	IIIB1	IIIB2	IIIB3

The Lemma IV cases will be in the table like the III cases. The Lemma V and VI cases are similar but not identical. We will see this later.

Lemma IV: For each rational $r<1$ there are six types of indiscernibles with $x_1>Q$, $x_1^r<x_2-x_1$ and $x_2-x_1<x_1^q$ for $q>r$. They are classified by:

A. $(x_3-x_1)>2(x_2-x_1)$

1. $\frac{x_i-x_1}{x_1^r}$ are type I

2. $\frac{x_i-x_1}{x_1^r}$ are type IIA

3. $\frac{x_i-x_1}{x_1^r}$ are type IIB

B. $(x_3-x_1)<2(x_2-x_1)$

1. $\frac{x_\omega-x_i}{x_\omega^r}$ are type I

2. $\frac{x_\omega-x_i}{x_\omega^r}$ are type IIA

3. $\frac{x_\omega-x_i}{x_\omega^r}$ are type IIB

Proof: Clearly any type is A or B. Suppose $\langle c_i\rangle$ realize IV(r)A. Then since $c_2-c_1>c_1^r$ we have $c_2-c_1>Hc_1^r$ for all $H\in Q$

as in the proof of Lemma III. So $\frac{c_2-c_1}{c_1^r} > Q$. Further, $(c_3-c_1) > H(c_2-c_1)$ for all H Q^+ and hence $\frac{c_3-c_1}{c_1^r} > H \frac{c_2-c_1}{c_1^r}$. So the $\frac{c_i-c_1}{c_1^r}$ are I, IIA or IIB indiscernibles. Hence a IV(r)A type must be in case A1, A2 or A3. Now suppose $\langle c_i : i \in N^+, c_\omega \rangle$ are of type IV(r)B. By an indiscernibly argument, $(c_3-c_1) < (1+h)(c_2-c_1)$ for all $h \in Q^+$. So $(c_3-c_2) < h(c_2-c_1) < h(c_3-c_1)$ for all $h \in Q^+$. Hence $H(c_\omega - c_2) < (c_\omega - c_1)$ for all $H \in Q^+$. Thus $H \frac{c_\omega - c_2}{c_\omega^r} < \frac{c_\omega - c_1}{c_\omega^r}$. So the $\frac{c_\omega - c_i}{c_\omega^r}$ are either I, IIA or IIB indiscernibles. Thus it remains to show that we get exactly one type for each of the six cases. As the proof is essentially the same as in Lemma III we will just list how IV(r) indiscernibles c_i's may be encoded into I-indiscernibles.

A1: $\left\langle \frac{c_i-c_1}{c_1^r}, c_1 \right\rangle$

A2: $\left\langle \frac{c_i-c_1}{c_2-c_1}, \frac{c_2-c_1}{c_1^r}, c_1 \right\rangle$

A3: $\left\langle \frac{c_\omega-c_1}{c_i-c_1}, \frac{c_\omega-c_1}{c_1^r}, c_1 \right\rangle$

B1: $\left\langle \frac{c_\omega-c_i}{c_\omega^r}, c_\omega \right\rangle$

B2: $\left\langle \frac{c_{\omega+1}-c_i}{c_{\omega+1}-c_\omega}, \frac{c_{\omega+1}-c_\omega}{c_{\omega+1}^r}, c_\omega \right\rangle$

B3: $\left\langle \frac{c_\omega-c_1}{c_\omega-c_i}, \frac{c_\omega-c_1}{c_\omega^r}, c_\omega \right\rangle$ □

<u>Lemma V</u>: For each irrational $r<1$ there are exactly four types with $x_1>Q$ and such that $x_1^s<x_2-x_1<x_1^t$ for $s<r<t$. They are classified as follows: Fix the $n\in N$ such that $0<n+r<1$, then

A. $(x_3-x_1)>2(x_2-x_1)$

1. $x_1^n(x_i-x_1)$ are type IIA
2. $x_1^n(x_i-x_1)$ are type IIB

B. $(x_3-x_1)<2(x_2-x_1)$

1. $x_\omega^n(x_\omega-x_i)$ are type IIA
2. $x_\omega^n(x_\omega-x_i)$ are type IIB

Proof: Clearly any type is in either case A or case B. Suppose $<c_i: i\in N^+>$ realizes V(r)A. Then since $(c_3-c_1)>2(c_2-c_1)$ we have $(c_3-c_1)>H(c_2-c_1)$ and hence $c_1^n(c_3-c_1)>Hc_1^n(c_2-c_1)$. Also, $c_1^n(c_2-c_1)>c_1^{n-r-\varepsilon}>H\ \forall H\in Q^+$. So the $c_1^n(c_i-c_1)$'s are either I, IIA or IIB indiscernibles. Now $c_1^n(c_3-c_1)<c_1^n c_1^{r+\varepsilon} = c_1^{n+r+\varepsilon}$. Also, $(c_1^n(c_2-c_1))^2>(c_1^n c_1^{r-\varepsilon})^2 = c_1^{2(n+r)-2\varepsilon}$, which for sufficiently small ε is $> c_1^{(n+r)+\varepsilon}$ so $(c_1^n(c_2-c_1))^2>c_1^n(c_3-c_1)$. Hence the $c_1^n(c_i-c_1)$'s cannot be type I. So any V(r) A type is in case A1 or case A2. Now suppose $<c_i:i\in N^+, c_\omega>$ are V(r) B indiscernibles. Then $H(c_3-c_2)<(c_3-c_1)$ by an indiscernibility argument as in Lemma III. So $H(c_\omega-c_3)<(c_\omega-c_2)$ for all $H\in Q^+$. Thus $c_\omega^n(c_\omega-c_i)$ are I, IIA or IIB. As above, the $c_\omega^n(c_\omega-c_i)$ cannot be of type I. So any V(r) B type must be in case B1 or B2. Hence, it remains to show that we get exactly one type for each case.

Case A1: Uniqueness: Suppose $<c_i>$ are V(r)A1 indiscernibles. Since $c_1^n(c_i-c_1)$'s are IIA we have that

$$\left\langle \frac{c_1^n(c_i-c_1)}{c_1^n(c_2-c_1)}, c_1^n(c_2-c_1)\right\rangle \quad \text{are I-indiscernibles}$$

i.e. $$\left\langle \frac{c_i-c_1}{c_2-c_1}, c_1^n(c_2-c_1)\right\rangle \quad \text{are I-indiscernibles.}$$

Call them d_i's, d_ω. Note that we cannot encode the c_i's with just these. However, we will use d_i's, d_ω and c_1 to encode the c_i's. We must show how any polynomial inequality in these can be resolved using only the case hypothesis. First any polynomial in the c_i's really is expressible as a polynomial in d_i's, d_ω and c_1. Suppose the monomials m_i in p are $q_i c_1^{a_i} d_\omega^{b_i} d_{i_1}^{e_{i_1}} \ldots d_{in}^{e_{in}}$. Then $|m_i| \approx c_1^{a_i} d_\omega^{b_i}$ because $d_\omega^{\varepsilon} > d_{i_n}$. Hence $|m_i| \approx c_1^{a_i+b_i(n+r)}$ because $d_\omega \approx c_1^n(c_2-c_1)$. Reorder the m_i's so that $a_1+b_1(n+r)$ is a largest approximate exponent. Let $J = \{i: m_i$ has approximate exponent $a_1+b_1(n+r)\}$. Say $m_1, m_2 \in J$, i.e., $a_1+b_1(n+r) = a_2+b_2(n+r)$. Then $a_1+b_1n - a_2-b_2n = (b_2-b_1)r$. Now since r is irrational and a_1, b_1, a_2, b_2, and n are rational we have $b_2-b_1=0$. Hence $b_2=b_1$ and $a_2=a_1$. So for all $i\in J$, m_i has c_1 exponent a_1 and d_ω exponent b_1. Now, $p>0 \leftrightarrow \sum_{i\in J} m_i > -\sum_{i\notin J} m_i \leftrightarrow c_1^{a_1} d_\omega^{b_1}(\sum\hat{m}) > -\sum_{i\notin J} m_i$ (where $\sum\hat{m}$ is $\sum_{i\in J} m_i/c_1^{a_i} d_\omega^{b_i}$). Also $|\sum_{i\notin J} m_i| < c_1^{a_1+b_1(n+r)-\varepsilon}$ for some ε, by the definition of J. So $c_1^{a_1} d_\omega^{b_1}(\sum\hat{m}) > \sum_{i\notin J} m_i \leftrightarrow \sum\hat{m} > 0$ (since $c_1^{a_1} d_\omega^{b_1} > |\frac{\sum m_i}{\sum\hat{m}}|$). Now since $\sum\hat{m}$ is a polynomial in I-indiscernibles, it is determined. So any polynomial inequality in the c_i's is completely determined, and we have uniqueness. Note that this procedure is recursive in r (i.e., if we can tell when $q_1>r$ and $r<q_2$ then we can effectively resolve any inequality).

Existence: Let $\langle d_i: i\in N^+\rangle$ be IIA-indiscernibles. Let e be such that $d_i^{1/q_1} < e < d_1^{1/q_2}$ for $0<q_1<n+r<q_2$. (This is possible by compactness and the conditions $n+r>0$ and $d_i \approx d_j$.) Let

$c_i = e + \frac{d_i}{e^n}$. (This is just the reverse of the coding given in the uniqueness part.) Then $c_2 - c_1 = (d_2-d_1)/e^n > \frac{d_1}{e^n} > \frac{e^{n+r-\varepsilon}}{e^n} = e^{r-\varepsilon}$. Now since $\frac{d_i}{e^n} \approx e^r$ we get $e + \frac{d_i}{e^n} \approx e$ and hence we have $e^{r-\varepsilon} > (e+\frac{d_i}{e^n})^{r-\hat{\varepsilon}}$ for $\hat{\varepsilon} > \varepsilon$. So $c_2-c_1 > c_1^{r-\hat{\varepsilon}}$. In other words $c_2-c_1 > c_1^q$ for all rational $q<r$. Also $c_2-c_1 = (d_2-d_1)/e^n < \frac{d_2}{e^n} < \frac{e^{n+r+\varepsilon}}{e^n} = e^{r+\varepsilon} < (e+\frac{d_1}{e^n})^{r+\varepsilon} = c_1^{r+\varepsilon}$. So provided the c_i's are indiscernibles they must be of type $V(r)$. Indeed $c_1^n(c_i-c_1) = \frac{d_i-d_1}{e^n}(e+\frac{d_1}{e^n})^n \approx d_i-d_1 \approx d_i$. So if the c_i's are indiscernibles, $c_1^n(c_i-c_1)$'s are IIA and we would have existence. Now we show that c_i's are indeed indiscernibles. Let us check that $\left\langle \frac{c_1^n(c_i-c_1)}{c_1^n(c_2-c_1)}, c_1^n(c_2-c_1)\right\rangle$ are I-indiscernibles. Note that if we had $c_1^n(c_i-c_1) = d_i$. This would be immediate, but now it requires a bit of work. Well for $i<j$ and $m \in N$

$$\left(\frac{c_1^n(c_i-c_1)}{c_1^n(c_2-c_1)}\right)^m < \left(\frac{2d_i}{\frac{1}{2}d_2}\right)^m = 4^m\left(\frac{d_i}{d_2}\right)^m < \frac{1}{4}\frac{d_j}{d_2} = \frac{\frac{1}{2}d_j}{2d_2}$$

$$< \frac{c_1^n(c_j-c_1)}{c_1^n(c_2-c_1)}. \quad \text{Also,} \quad \left(\frac{c_1^n(c_i-c_1)}{c_1^n(c_2-c_1)}\right)^m < \left(\frac{2d_i}{\frac{1}{2}d_2}\right)^m < \tfrac{1}{2}d_2 < c_1^n(c_2-c_1).$$

So by the proof of Lemma I they are I-indiscernibles. Indeed we have $\left\langle \frac{c_{i_j}-c_{i_1}}{c_{i_2}-c_{i_1}} \text{'s},\ c^n_{i_1}(c_{i_2}-c_{i_1}) \right\rangle$ are I-indiscernibles

and $c_{i_2}-c_{i_1} \approx c^r_{i_1}$ whenever $i_1<i_2\ldots<i_n$ by the same argument. These conditions are all we need to apply the algorithm given in the uniqueness proof. Since the conditions hold for any increasing sequence of c_i's we have that the c_i's are indiscernibles, and hence V(r)A1 indiscernibles.

Case A2: Uniqueness: Suppose $<c_i: i\in N^+, c_\omega>$ are V(r)A2 indiscernibles. Then $c^n_1(c_i-c_1)$ are IIB and hence $\left\langle \frac{c^n_1(c_\omega-c_1)}{c^n_1(c_i-c_1)},\ c^n_1(c_i-c_1) \right\rangle$ are I-indiscernibles and

$c^n_1(c_i-c_1) \approx c^{n+r}_1$. With this encoding we can apply the algorithm given in case A1, to determine all polynomial inequalities.

Existence: Let d_i's be IIB indiscernibles and choose e such that $d_1^{1/q_1} < e < d_1^{1/q_2}$ for $0<q_1<n+r<q_2$. Let $c_i = e + \frac{d_i}{e^n}$. The argument for case A1 can be slightly modified to show that these c_i's are V(r)A2 indiscernibles.

Case B1: Uniqueness: Let $<c_i\text{'s}, c_\omega>$ be V(r)B1. Then the $c^n_\omega(c_\omega-c_i)$'s are IIA so $\left\langle \frac{c_\omega-c_i}{c_\omega-c_1}\text{'s},\ c^n_\omega(c_\omega-c_1) \right\rangle$ are type I and $c_\omega^n(c_\omega - c_i) \approx c_\omega^{n+r}$. With this encoding we can apply the algorithm in case A1 to determine all inequalities.

Existence: Let $\langle d_\omega, d_i : i \in (N^+)^* \rangle$ be IIA indiscernibles and let e be such that $d_1^{1/q_1} < e < d_1^{1/q_2}$ for $0<q_1<n+r<q_2$.

Let $c_i = e - \frac{d_i}{e^n}$. $c_\omega^n(c_\omega - c_i) = \frac{d_i - d_\omega}{e^n}(e + \frac{d_i}{e^n})^n \approx d_i - d_\omega \approx d_i$.

Also $c_2 - c_1 \approx c_1^r$. So by essentially the same argument as in case A1, the c_i's are V(r) B1 indiscernibles.

Case B2: Uniqueness: The proof is like that of Case B1, but now $\left\langle \frac{c_{\omega+1} - c_\omega}{c_{\omega+1} - c_i}, c_{\omega+1}^n(c_{\omega+1} - c_\omega) \right\rangle$ are I-indiscernibles and $c_{\omega+1}^n(c_{\omega+1} - c_i) \approx c_{\omega+1}^{n+r}$.

Existence: As in Case B1, but with d_i's chosen as IIB indiscernibles. □

<u>Lemma VI</u>: There are exactly ten types of indiscernibles with $x_1 > Q$ and with $x_2 - x_1 < x_1^{-n}$ for all $n \in N$. They are classified by:

A. $(x_3 - x_1) > 2(x_2 - x_1)$

1. $\frac{1}{x_i - x_1}$ are type I

2. $\frac{1}{x_i - x_1}$ are type IIA

a) $\frac{x_3 - x_1}{x_2 - x_1} < x_1$

b) $x_1 < \frac{x_3 - x_1}{x_2 - x_1}$

3. $\frac{1}{x_i - x_1}$ are type IIB

a) $\frac{x_3 - x_1}{x_2 - x_1} < x_1$

b) $x_1 < \frac{x_3 - x_1}{x_2 - x_1}$

B. $(x_3 - x_1) > 2(x_2 - x_1)$

1. $\frac{1}{x_\omega - x_i}$ are type I

2. $\frac{1}{x_\omega - x_i}$ are type IIA

a) $\frac{x_3 - x_1}{x_3 - x_2} < x_3$

b) $x_3 < \frac{x_3 - x_1}{x_3 - x_2}$

3. $\frac{1}{x_\omega - x_i}$ are type IIB

a) $\frac{x_3-x_1}{x_3-x_2} < x_3$

b) $x_3 < \frac{x_3-x_1}{x_3-x_2}$.

Proof: Clearly any type is either in Case A or Case B. Suppose $<c_i>$ realizes Case VIA. Then $c_1^n < \frac{1}{c_i-c_1}$, so $\frac{1}{c_i-c_1} > Q$. Also $(c_3-c_1)>2(c_2-c_1)$ so $\frac{1}{c_2-c_1} > 2\frac{1}{c_3-c_1}$. Hence the $\frac{1}{c_i-c_1}$ are I, IIA or IIB. Note that 2a, 2b partition subcase 2 and similarly 3a, 3b partition subcase 3. So the outline is a partition of the A cases. Suppose $<c_i: i\in N^+, c_\omega>$ is type VIB. Then $H(c_3-c_2)<(c_3-c_1)$, so $H\frac{1}{c_\omega-c_1} < \frac{1}{c_\omega-c_2}$ for all $H\in Q^+$. $Q < \frac{1}{c_\omega-c_1}$, so the $\frac{1}{c_\omega-c_i}$'s are I, IIA or IIB. So again we get a partition of the cases. So it will suffice to show uniqueness and existence for each of the ten cases. Uniqueness is shown by coding the c_i's into I-indiscernibles. Existence is shown by undoing the coding from the uniqueness part. Now if in case A we have $c_1^n < \frac{1}{c_2-c_1}$ and if in Case B we have $c_\omega^n < \frac{1}{c_\omega-c_1}$. This is what gives us the extra subcases for A2, A3, B2 and B3. Since the proofs are essentially those of the previous lemmas we will just give the coding into I-indiscernibles.

A1: $\langle c_1, \frac{1}{c_i - c_1} \rangle$

A2a: $\langle \frac{c_\omega - c_1}{c_i - c_1}, c_1, \frac{1}{c_\omega - c_1} \rangle$

(To uncode, let the d_i's be IIA indiscernibles and choose e such that $(\frac{d_i}{d_\omega})^n < e$ and $e^n < d_\omega$; then let $c_i = \frac{1}{d_i} + e$) .

A2b: $\langle c_1, \frac{c_\omega - c_1}{c_i - c_1}, \frac{1}{c_\omega - c_1} \rangle$

A3a: $\langle \frac{c_i - c_1}{c_2 - c_1}, c_1, \frac{1}{c_2 - c_1} \rangle$

A3b: $\langle c_1, \frac{c_i - c_1}{c_2 - c_1}, \frac{1}{c_2 - c_1} \rangle$

B1: $\langle c_\omega, \frac{1}{c_\omega - c_i} \rangle$

B2a: $\langle \frac{c_\omega - c_1}{c_\omega - c_i}, c_\omega, \frac{1}{c_\omega - c_1} \rangle$

B2b: $\langle c_\omega, \frac{c_\omega - c_1}{c_\omega - c_i}, \frac{1}{c_\omega - c_1} \rangle$

B3a: $\langle \frac{c_{\omega+1} - c_i}{c_{\omega+1} - c_\omega}, c_{\omega+1}, \frac{1}{c_{\omega+1} - c_\omega} \rangle$

B3b: $\langle c_{\omega+1}, \frac{c_{\omega+1} - c_i}{c_{\omega+1} - c_\omega}, \frac{1}{c_{\omega+1} - c_\omega} \rangle$ □

Now we have classified all the types of indiscernibles with x_1

infinite. Indeed we have shown more. We have shown how we can transform most of the types into an I type. But let us first summarize the primary result:

<u>Theorem 3.2</u>: The following is a complete classification of all the types of indiscernibles containing $x_1 > Q$.

I. $x_1^2 < x_2$

II. $2x_1 < x_2 < x_1^2$

A. $\left(\frac{x_2}{x_1}\right)^2 < \left(\frac{x_3}{x_1}\right)$

B. $\left(\frac{x_3}{x_1}\right) < \left(\frac{x_2}{x_1}\right)^2$

III(r) (for each r rational ≤ 1) $x_2 - x_1 < x_1^r$ and for all $q<r$ $x_1^q < x_2 - x_1$.

A. $2(x_2-x_1)<(x_3-x_1)$

1. $\frac{x_1^r}{x_i-x_1}$ are type I

2. $\frac{x_1^r}{x_i-x_1}$ are type IIA

3. $\frac{x_1^r}{x_i-x_1}$ are type IIB .

B. $(x_3-x_1)<2(x_2-x_1)$

1. $\dfrac{x_\omega^r}{x_\omega-x_i}$ are type I

2. $\dfrac{x_\omega^r}{x_\omega-x_i}$ are type IIA

3. $\dfrac{x_\omega^r}{x_\omega-x_i}$ are type IIB

IV(r) (for each r rational < 1) $x_1^r<x_2-x_1$ and for all $q>r$ $x_2-x_1<x_1^q$.

A. $2(x_2-x_1)<(x_3-x_1)$

1. $\dfrac{x_i-x_1}{x_1^r}$ are type I

2. $\dfrac{x_i-x_1}{x_1^r}$ are type IIA

3. $\dfrac{x_i-x_1}{x_1^r}$ are type IIB

B. $(x_3-x_1)<2(x_2-x_1)$

1. $\dfrac{x_\omega-x_i}{x_\omega^r}$ are type I

2. $\frac{x_\omega - x_i}{x_\omega^r}$ are type IIA

3. $\frac{x_\omega - x_i}{x_\omega^r}$ are type IIB

V(r) (for each irrational $r<1$) for all s_1

Let $n \in N$ be such that $0<n+r<1$

A. $2(x_2-x_1)<(x_3-x_1)$

1. $x_1^n(x_i-x_1)$ are the type IIA

2. $x_1^n(x_i-x_1)$ are the type IIB

B. $(x_3-x_1)<2(x_2-x_1)$

1. $x_\omega^n(x_\omega-x_i)$ are type IIA

2. $x_\omega^n(x_\omega-x_i)$ are type IIB

VI for all $n \in N$ $x_2-x_1<x_1^{-n}$

A. $2(x_2-x_1)<(x_3-x_1)$

1. $\frac{1}{x_i - x_1}$ are type I

2. $\frac{1}{x_i - x_1}$ are type IIA

a) $\frac{x_3 - x_1}{x_2 - x_1} < x_1$

b) $x_1 < \frac{x_3 - x_1}{x_2 - x_1}$

3. $\frac{1}{x_i - x_1}$ are type IIB

a) $\frac{x_3 - x_1}{x_2 - x_1} < x_1$

b) $x_1 < \frac{x_3 - x_1}{x_2 - x_1}$

B. $(x_3 - x_1) < 2(x_2 - x_1)$

1. $\frac{1}{x_\omega - x_i}$ are type I

2. $\frac{1}{x_\omega - x_i}$ are type IIA

a) $\frac{x_3 - x_1}{x_3 - x_2} < x_3$

b) $x_3 < \frac{x_3 - x_1}{x_3 - x_2}$

3. $\frac{1}{x-x_i}$ are IIB

a) $\frac{x_3-x_1}{x_3-x_2} < x_3$

b) $x_3 < \frac{x_3-x_1}{x_3-x_2}$

Proof: By above. □

Note: Although there are only six general cases, the cases III, IV or V are parameterized by $r \in A$ (for 3 different sets A) so that there are actually 2^ω types with $x_1 > Q$.

<u>Corollary to Proof 3.2:</u> If c_i's are infinite indiscernibles over ϕ, then they are also indiscernible over R.

Proof: If the c_i's are not of type V then we may encode them as I-indiscernibles. The algorithm of Lemma I works over the coefficient domain R as well as for Q (i.e. find the sign of the largest monomial). The algorithm is independent of the indices so the c_i's are indiscernible over R. If the c_i's are of type V(r) we may encode them as I-indiscernibles d_i's and e where $d_i^{1/(n+r)} \approx e$. The algorithm of Lemma V also works over the coefficient domain R. So in this case we also have the c_i's indiscernible over R. □

We now look at the positive finite indiscernibles.

Corollary 3.3: The following classifies all indiscernible types with $x_1>0$ and with $x_1<n$ for some $n\in N$.

(i)$_r$ (for each algebraic number $r>0$) $r<x_1$ and for all $s\in Q$ with $s>r$, $x_1<s$. Subcases X determining the type of $\frac{1}{x_i-r}$'s. We get all possible infinite types.

(ii)$_r$ (for each algebraic $r>0$) $x_1<r$ and for all $s<Q$ with $s<r$, $s<x_1$. Subcases X determining the type of $\frac{1}{r-x_i}$'s. We get all possible infinite types.

(iii)$_r$ (for each non-algebraic $r>0$) for all $s,t\in Q$ with $s<r<t$, $s<x_1<t$.

A. $2(x_2-x_1)<(x_3-x_1)$

1. $\frac{1}{x_i-x_1}$ are type I

2. $\frac{1}{x_i-x_1}$ are type IIA

3. $\frac{1}{x_i-x_1}$ are type IIB

B. $(x_3-x_1)<2(x_2-x_1)$

1. $\frac{1}{x_\omega-x_i}$ are type I

2. $\frac{1}{x_\omega - x_i}$ are type IIA

3. $\frac{1}{x_\omega - x_i}$ are type IIB

Proof: Note that technically algebraic numbers are not in the language of RCF, but since they are definable we can use them. Now if c_i's are positive finite indiscernibles then they are obviously $(i)_r$, $(ii)_r$ or $(iii)_r$ for some $r \in R$. We show that the subcases are as stated in the outline.

Case $(i)_r$: Uniqueness: Suppose $\langle c_i \rangle$ satisfy $(i)_r$. Let $d_i = \frac{1}{c_i - r}$. The d_i's are infinite indiscernibles since the c_i's are indiscernible over the definable number r. Suppose the d_i's are type X. Then for any polynomial P, $p(\vec{c}_i)>0 \leftrightarrow p(\frac{\vec{1}}{d_i}+r)>0 \leftrightarrow q(\vec{d}_i,r)>0$ for the natural $q \leftrightarrow \vee\wedge \pm \hat{q}_j(\vec{d}_i)>0$ by using QE to eliminate the r. Now $\hat{q}_j(\vec{d}_i) \lesseqgtr 0$ is determined by the type X of the d_i's and hence $p(\vec{c}) \gtrless 0$ is determined. Thus each case X of d_i's specifies at most one type.

Existence: Let d_i's be infinite indiscernibles of type X. Let $c_i = \frac{1}{d_i} + r$. The d_i's are indiscernible over r, because r is definable. Hence the c_i's are indiscernibles. Indeed they are $(i)_r$ of Case X because $\frac{1}{c_i - r} = d_i$ which are of type X.

Case $(ii)_r$: As above, but let $d_i = \frac{1}{r-c_i}$.

Case $(iii)_r$: Surely any $<c_i>$'s realizing $(iii)_r$ are Case A or Case B. If the c_i's realize A then $\frac{1}{c_i-c_1}$ are I, IIA or IIB because $H(c_2-c_1)<(c_3-c_1)$. If the c_i's realize B then $\frac{1}{c_\omega-c_i}$ are I, IIA or IIB because $H(c_3-c_2)<(c_3\ c_1)$. So it will suffice to show we get at most one type for each subcase.

Uniqueness:

A Cases: Let $<c_i>$ be of type $(iii)_r$. Then $\frac{1}{c_i-c_1}$ are I, IIA or IIB and so may be encoded as I-indiscernibles d_i. So any $p(c_i)>0 \leftrightarrow q(d_i,c_1)>0$. Now $Q(c_1) \cong Q(r)$ because r is non-algebraic and $c_1 \approx r$. Indeed $Q(c_1,\vec{d}_i) \cong Q(r,\vec{d}_i)$. So $q(d_i,c_1)>0 \leftrightarrow q(d_i,r)>0$. By Corollary 3.2, the d_i's are indiscernible over R. Hence $q(d_i,r)\gtrless 0$ is determined so the type of $\frac{1}{c_i-c_1}$ fixes the type of the c_i's.

B Cases: As above, but now $\frac{1}{c_\omega-c_i}$ are the d_i's and $Q(\omega) \cong Q(r)$.

Existence:

A Cases: Let d_i's be indiscernibles of type I, IIA or IIB. Let $c_i = \frac{1}{d_i} + r$. Since the d_i's are infinite indiscernibles, they are indiscernible over r. Hence the c_i's are indiscernibles. Also the c_i's are $(iii)_r$. Now $\frac{1}{c_i-c_1} = \frac{1}{\frac{1}{d_i} - \frac{1}{d_1}}$ =

$$\frac{1}{\frac{d_1-d_i}{d_1 d_i}} = \frac{d_1 d_i}{d_1-d_i} \approx d_i. \text{ So the } \frac{1}{c_i-c_1} \text{ are I, IIA or IIB}$$

depending on the d_i's.

B Cases: Let d_i's be indiscernibles of type I, IIA or IIB. Let $c_i = r - \frac{1}{d_i}$. The c_i are $(iii)_r$ indiscernibles as above. Also,

$$\frac{1}{c_\omega - c_i} = \frac{1}{\frac{1}{d_i} - \frac{1}{d_\omega}} = \frac{1}{\frac{d_\omega - d_i}{d_\omega d_i}} = \frac{d_i d_\omega}{d_\omega - d_i} \approx d_i. \text{ So the } \frac{1}{c_\omega - c_i}$$

are I, IIA or IIB depending on the d_i's. □

<u>Corollary 3.4</u>: The types containing $x_1<0$ are completely classified by specifying the type of $-x_i$'s.

Proof: Obvious. □

Thus we have a complete classification of all the order indiscernibles of RCF. A couple of interesting consequences are the following:

<u>Corollary to Proof 3.5</u>: A type of indiscernibles is recursive in two extended reals, r and s, where s is the cut in Q of x_1 and r is such that for

$$s = +\infty: \quad x_2 \approx x_1 + x_1^r$$

$$s = -\infty: \quad -x_2 \overset{\sim}{\sim} -x_1 + (-x_1)^r$$

$$s \text{ finite}, x_1 > s: \quad \frac{1}{x_1 - s} \overset{\sim}{\sim} \frac{1}{x_2 - s} + \left(\frac{1}{x_2 - s}\right)^r$$

$$s \text{ finite}, x_1 < s: \quad \frac{1}{s - x_2} \overset{\sim}{\sim} \frac{1}{s - x_1} + \left(\frac{1}{s - x_1}\right)^r .$$

Proof: We just use the fact that we can code up the indiscernibles and then apply either Lemma I or Lemma V to determine any inequality. (Note: at most one of r and s is non-recursive.)

Corollary 3.6: A type of an indiscernible set is uniquely determined by the polynomial inequalities in four variables.

Proof: Just check how many variables are needed to specify the conditions in the outlines.

B. Interdefinability

As in the case of DOAG's we have interdefinability between many of the types, but this time not between all of them.

Definition: A type of indiscernibles with r rational, $+\infty$, or $-\infty$ and with s algebraic, $+\infty$, or $-\infty$ (as in Corollary 3.5) will be called algebraic. Otherwise it will be called nonalgebraic.

Also we will use the expression I type to mean the $r=+\infty$, $s=+\infty$ type.

<u>Corollary 3.7</u>: If $<c_i: i\epsilon Q>$ is an algebraic sequence of indiscernibles then there is a definable map over a finite number of the c_i's taking a subset of the c_i's of index set isomorphic to Q onto indiscernibles $<d_j: j\epsilon Q>$ of any other algebraic type.

<u>Proof Sketch</u>: Apply the codings given in the uniqueness proofs to convert c_i's into e_k's of type I. Then use an inverse coding to get the e_k's to map onto the d_j's. Of course this only works smoothly if the index sets are appropriate for the codings. To handle this difficulty we may have to throw out some of the c_i's. For example, we show IIB indiscernibles are definable from IIA indiscernibles. Let $<c_i: i\ Q>$ be IIA. Take a subsequence ordered as 1+Q, and relabel these c_i's as $<c_o, c_i: i\epsilon Q>$. $\left\langle \frac{c_i}{c_o} : i\epsilon Q,\ c_o \right\rangle$ are I-indiscernibles. Call them $<e_i: i\epsilon Q,\ e_\omega>$. Now by inverting the coding for IIB we see that $\left\langle \frac{e_\omega}{e_i},\ e_\omega \right\rangle$ are IIB indiscernibles. Hence, the map $f: \frac{c_o^2}{c_i} \to d_i$ is the desired coding. The other cases work in essentially the same way. □

<u>Corollary 3.8</u>: If an RCF contains algebraic indiscernibles $<c_i: i\epsilon Q>$ then it contains algebraic indiscernibles $<d_i: i\epsilon Q>$

of every algebraic type.

Proof: Immediate by Corollary 3.7. □

However, the nonalgebraic types are a different matter.

Proposition 3.9: The RCF F generated by the type I indiscernibles $<c_i: i\epsilon I>$ contains no nonalgebraic indiscernibles.

Proof: Any element in F is the solution of some $\Sigma p_i(\vec{c})x^i$. Suppose a solution is $e\approx r\epsilon R$, where r is a transcendental. Then $\Sigma p_i(\vec{c})r^i\approx 0$. Say m_1 is the largest monomial occurring in $\Sigma p_i(\vec{c})r_i$. Then $\Sigma q_i r^i \approx 0$ where q_i is the coefficient of m_1 in $p_i(\vec{c})$. But this can't happen because r is transcendental. So there are no s=nonalgebraic types in F. Now suppose e is infinite and a solution to $\Sigma p_i(\vec{c})x^i$. It is easy to check that e cannot be a new I-indiscernible, so $e\sim c_i^r$ for some r. Indeed since all the monomials must cancel out r must be rational. In particular if $<d_i>$ is a sequence of infinite indiscernibles with r<1 then $d_1\approx d_i\sim c_i^r$ for some r rational. Also, $d_2-d_1\sim c_j^s$ for some s rational. But then the d_i's cannot be of type V. Hence there are no r=irrational, s=+∞ indiscernibles in F. Hence there can be no r=irrational indiscernibles in F. So F contains no non-algebraic indiscernibles. □

So in looking at indiscernibility types in models we want to focus primarily on the I-type and also on the nonalgebraic types.

C. Structure Theory

Proposition 3.10: A RCF F has up to order isomorphism a unique maximal index set of type I-indiscernibles.

Proof: By Zorn's Lemma there exists maximal sets. Suppose $\langle c_i : i \in I \rangle$ and $\langle d_j : j \in J \rangle$ are both maximal in F. In the language of ordered groups $\{2, c_i : i \in I\}$ and $\{2, d_j : j \in J\}$ are DOAG (1) indiscernibles (since for all n, 2^n is finite). Indeed they are maximal in F. So by Proposition 2.5 we have $(I,<) \cong (J,<)$. □

Indeed we can say a bit more.

Theorem 3.11: If $(F,<)$ is an ordered field containing $\langle c_i : i \in I \rangle$ as a maximal set of I-indiscernibles, then $\langle c_i : i \in I \rangle$ is also maximal for the real closure of F.

Proof: Pick any e in the real closure of F. We will show that e is not a new c_i. We may assume $e>1$ by taking negatives or reciprocals. Now we have $a_n e^n + \ldots a_1 e + a_0 = 0$ for some a_i's in F. For each i, if a_i is infinitesimal (i.e. $< \frac{1}{n}$ for all $n \in N^+$) multiply the polynomial by $\frac{1}{a_i}$. This guarantees at least one coefficient is infinite or finite. Suppose $|a_n|$ is the largest coefficient. If a_k is finite then the polynomial is "near standard" (i.e. close to a polynomial over that reals). Then e must also be near standard (i.e. $e \approx r$, for some $r \in R$). Thus e would not be a new c_i. So suppose $|a_n| \approx c^{\alpha_i}$ were $\alpha_i \in R$. Let $a_{i_1}, a_{i_2} \ldots a_{i_k}$ be all the a_i's such that $|a_{i_j}| \approx c_i^{\alpha_i}$. Say we have $\hat{n}$ such a_{i_j}'s.

Case 1: $\hat{n}=1$. Let $|a_j|$ be the second largest coefficient. If $|a_j|^n < |a_k|$ for all $n \in N$ then we will get $e \approx c_i^{\alpha}$ for some α. For if $e < |a_k|^{\varepsilon}$ for all $\varepsilon \in Q^+$ then the $a_k e^k$ term will dominate and the polynomial could not be zero. While if $e > |a_k|^n$ for all $n \in N$ then the $a_n e^n$ term will dominate. Thus $e \approx c_i^{\alpha}$ and we would be done. So assume $|a_j| \approx |a_k|^{\alpha'}$ $(\alpha'<1)$. Multiply the polynomial by $\frac{1}{a_j}$ and we get

$$\sum \frac{a_\ell}{a_j} e^{\ell} + \sum \frac{a_{\hat{\ell}}}{a_j} e^{\hat{\ell}} + \frac{a_k}{a_j} e^k = 0$$

where $\frac{a_\ell}{a_j}$'s are finite, $\frac{a_{\hat{\ell}}}{a_j}$'s are infinitesimals, and $\frac{a_k}{a_j} \approx c_i^{\alpha}$, where $\alpha = \frac{1-\alpha'}{\alpha_i} > 0$. If $e > c_i^n$ for all n then $a_n e^n$ will dominate, a contradiction. If $e < c_i^{\varepsilon}$ for all ε, then $\frac{a_k}{a_j} e^k$ will dominate which is also a contradiction. So $e \approx c_i^a$ for some $a \in R$ and we are done with case 1.

Case 2: $\hat{n}>1$. Let $\hat{a}$ be the smallest of the $a_{i_1}\ldots a_{i_k}$'s with $|a_{i_j}| \approx c_i^{\alpha_i}$. Multiply the polynomial by $\frac{1}{\hat{a}}$ and we get

$$\sum_{i\in I_1} b_{1i}e^i + \sum_{i\in I_2} b_{2i}e^i + \sum_{i\in I_3} b_{3i}e^i = 0$$ with b_{1i}'s infinite,

the b_{2i}'s finite and the b_{3i}'s infinitesimal. We have at least one $i\in I_2$ (since $\frac{a}{\hat{a}} = 1$) and at least one other infinite or finite b_i (namely $\frac{a_n}{\hat{a}}$). We may assume $e < |a_k|^\varepsilon$ for all ε, or else we are done. Hence $e < |\hat{a}|^\varepsilon$ for all $\varepsilon\in Q^+$. Then, for each $\ell\in I_3$, $b_{3\ell}e^\ell = \frac{a_\ell}{\hat{a}}e^\ell$ which is an infinitesimal (because $|a_k| < c^{\alpha_i-\varepsilon}$, so $|\frac{a_\ell}{\hat{a}}| < \frac{1}{d_i^\varepsilon}$ and $e<d_i^{\varepsilon'}$ for $\varepsilon'<\varepsilon$). So we will be able to ignore these terms. We now repeat the process by applying either Case 1 or Case 2.

Eventually this process will terminate because every application of Case 2 reduces the number of non-infinitesimal terms by at least one, and yet retains at least one non-infinitesimal coefficient (in fact at least 2). Also, the "ignored terms" stay infinitesimal--indeed they get smaller. So eventually we have $e \approx r\in R$ or $e \approx \left(\frac{a_k}{a_{i_1}\ldots a_{i_j}}\right)^\alpha$. So e is not a new d_i. □

<u>Corollary 3.12</u>: Any ordered field $(F,<)$ has a unique index set for a maximal set of I-indicernibles, up to order isomorphism.

Proof: Apply Proposition 3.11 and Theorem 3.12. □

Let us now take a look at some of the RCF's, F, for a fixed I. Well, I=ϕ iff F is a real closed subfield of R. What about for I = {1}? The smallest such subfield is $\overline{Q(c)}$ (the real closure of Q(c)) with c>Q. A larger field is $\overline{F_o(c)}$ for $F_o \subseteq R$. In fact, we can extend even further. Let L be the Levi-Civita nonarchimedian field. I.e., $L = \{a_o t^{v_o} + a_1 t^{v_1} + \ldots : a_i \in R,\ v_i \in R$ with the v_i increasing and unbonded, and t an infinitesimal }. Note that $\{\frac{1}{t}\}$ is a maximal set for L. More generally, for any I the minimal model with maximal I-indiscernibles $\langle c_i : i\in I\rangle$ is the field $\overline{Q(c_i : i\in I)}$. The maximal ones are formal power series fields.

Definition: The formal power series field, F[[G]] where F is a field and G is an ordered abelian group, is the set $W(F_a, a\in(G,<))$ with F_a=F and with $(\alpha+\beta)_a = \alpha_a+\beta_a$ and $(\alpha\cdot\beta)_a = \sum_{bc=a} \alpha_b \beta_c$.

Note that since α and β are well founded sequences the product is well defined. By Hahn, F[[G]] is an ordered field (see Fuchs). Also, any F is embeddable in R[[G]] where G is the ordered group of archimedean classes of F (see Fuchs, but note that he does things in a more general fashion.) Now if F has maximal indiscernible set $\langle c_i : i\in I\rangle$ then its group of archimedean classes G is contained in $W(R_i: i\epsilon I^+)$ where $R_i = R$ and $I^+ = <\{0\},I>$. (Recall that $<2,c_i: i\epsilon I>$ is a maximal (1)-indiscernible set as a DOAG, which is why we use I^+.) Hence, F embeds in $R[[W(R_i:i\epsilon I^+)]]$. Also $R[[W(R_i: i\epsilon I^+)]]$ has archimedean classes $W(R_i:i\epsilon I)$, so it has maximal indiscernibles $<c_i:i\epsilon I>$. Hence the maximal ordered field F with index set I is $R[[W(R_i:i\epsilon I^+)]]$. (Note by maximality and Proposition 3.11 F is real closed.) So any F with maximal index set $<c_i: i\epsilon I>$ satisfies: $\overline{Q(c_i: i\epsilon I)} \subseteq F \subseteq R[[W(R_i: i\epsilon I^+)]]$.

The use of the other algebraic indiscernible types yields no new information. If we use the non-algebraic types we do get slightly more information. First the collection of types with s irrational indicates the residue field. That is, let $A = \{s: s$ occurs as an (r,s) type in $F\}$. Then $\overline{A(c_i: i\epsilon I)} \subseteq F \subseteq A[[R_i: i\epsilon I^+]]$. We also know a little about the component fields A_i. But since the choice of the c_i's is not canonical we can't really pin down what the A_i are. There is the further problem that the types don't tell which irrational powers occur in the same field A_i. Finally, we can't determine which infinite sequences occur in F. That is, we cannot tell how pseudo-complete (spherically complete) that F is. So the classification of RCF indiscernibles is not as comprehensive as for DOAG's. But given that there is so much flexibility as to how an ordered field may be extended, it is relatively nice.

4. OTHER EXAMPLES

The preceding two chapters show how order indiscernibles may be used to partially characterize models for the theories RCF and DOAG. The following examples are not technically difficult, but they are presented to provide more information as to how well the ordered indiscernibles can be used to describe models.

A. Variations on Dense Linear Orders

The following is a generalized version of dense linear orders. Let K be any index set. Let $L_K = \{<, P_\alpha(x), P_\alpha^<(x) : \alpha \in K\}$. Let T_K be the theory:

(1) $<$ is a dense linear order without endpoints

(2) The P_α's are infinite

(3) $x \in P_\alpha \wedge y \in P_\beta \to x<y$ (for every $\alpha<\beta$)

(4) $x \in P_\alpha \wedge z \in P_\alpha \wedge x<y<z \to y \in P_\alpha$

(5) $P_\alpha^<(x) \leftrightarrow \forall z (P_\alpha(z) \to z<x)$

(6) $P_\alpha^<(x) \to (P_\beta(x) \vee P_\beta^<(x))$ for every $\alpha<\beta$ such that $\forall \gamma\ (\gamma \leq \alpha \vee \beta \leq \gamma)$

We may think of the P 's as naming skies or pieces of a dense linear order. Note that T_K has QE. (This is the reason for the $P^{<}_{\alpha}$'s.) Let M = {Proper Dedekind cuts of K which are left and right limit points}, L = {Proper Dedekind cuts of K which are left limit points} and let R = {Proper Dedekind cuts of K which are right limit points but not left limit points}.

Proposition 4.1: The indiscernible types for T_K are

$(1)_{\alpha,\alpha\in K}$, $x_1 \in P_\alpha$.

$(2)_{m,m\in M}$ $P^{<}_{\alpha}(x_1) \wedge \sim P_\beta(X_1)$ for every α,β with $\alpha<m<\beta$.

$(3)_{\ell,\ell\in L}$ $P^{<}_{\alpha}(x_1) \wedge \sim P^{<}_{\ell+1}(x_1) \wedge \sim P_{\ell+1}(x_1)$ for every $\alpha<\ell$

where $\ell+1$ is the first point of K bigger than ℓ.

$(4)_{r,r\in R}$ $P^{<}_{r-1}(x_1) \wedge \sim P^{<}_{\alpha}(x_1)$ for all $\alpha>r$, where $r-1$

is the first point of K less than r.

Proof: For existence use compactness and for uniqueness use automorphisms. □

So for complicated K's there is a natural yet complicated classification of the types. Also note that the collection of maximal sets of indiscernibles completely specifies a model up to isomorphism.

B. Atomless Boolean Algebras

For the definition of this theory see for instance Chang and Keisler.

<u>Proposition 4.2</u>: There are 5 types of indiscernibles for Atomless Boolean Algebras.

(1) $x_1 \cap x_2 = 0 \wedge x_1^c \cap x_2 \neq 0 \wedge x_1^c \cap x_2 \neq 0 \wedge x_1^c \cap x_2^c \neq 0$

(2) $x_1 < x_2$

(3) $x_2 < x_1$

(4) $x_1 < x_2^c$

(5) $x_1^c < x_2$

Proof: Uniqueness: If $\langle c_i : i \in I \rangle$ are indiscernibles which satisfy (1), then the c_i's are independent and hence all formulas are determined. Otherwise we must be in one of Cases (2)-(5). We show each of these is unique. Well if $c_1 < c_2$ the c_i's must be linearly ordered so the type is specified. Similarly, with Case (3). If $c_1 < c_2^c$ then $c_2 < c_1^2$ and hence $c_3 < c_1^c$ by indiscernibility. Thus $c_3 < c_2^c$ also. Hence the c_1's are an anti-chain and the type is specified. Case (5) is like Case (4) only now the c_i^c's are an antichain.

Existence: Use the compactness theorem. □

Now types (2) and types (3) are essentially the same even though they are not explicitly interdefinable in the theory. Types (4) and types (5) are interdefinable by $x_i \to x_i^c$. So we have only 3 kinds of maximal indiscernible types to describe a model. Clearly these are important, but they are not nearly enough to characterize a model up to isomorphism. Note that this example also shows that a theory can be "independent" (i.e., have 2^κ 1-types over κ constants) and still have a simple classification of its indiscernible types.

C. Real Vector Spaces of Dimension 1

We now look at an example closely related to RCF--the theory of one dimensional vector spaces over a real closed field. Since the models of this theory are "isomorphic" to models of RCF we ought to get a similar classification for the indiscernibles. Let $L = \langle V, +_V, 0_V, R, +_R, \cdot_R, 0_R, 1_R, <_R, \cdot_{RV} \rangle$ and let

$T = R$ is an RCF + V is a vector space over R +

$$\forall y \in V \ (y \neq 0 \to \forall x \in V \ \exists r \in R (x = ry))$$

Let $f(x,y) = \begin{cases} 0 & \text{if } y=0 \\ r & \text{for the unique } r \text{ s.t. } x=ry \text{ if } y\neq 0. \end{cases}$

Then T has QE over the language $L\cup\{f\}$. The possible types with $x\in R$ are exactly those for an RCF. If $x\in V$ then things are a little more complicated. Well, if some $v_o\neq 0$ was added as a constant to the language then $c_i = f(c_i,v_o)v_o$ and so the type of c_i would be fixed by the type of $f(c_i,v_o)\in R$. But we have assumed that there is no $v_o\in L$ so there are potentially fewer types.

<u>Proposition 3.3</u>: The types of indiscernibles $<c_i>$ with $c_i\in V$ are specified by the type of $\frac{e_i}{e_1}$ where $c_i = \frac{e_i}{e_1} c_1$ and the e_i's are indiscernibles in RCF.

Proof: Pick any indiscernibles $<c_i>$ such that $c_1\in V$. Let a_i be such that $c_i = a_i c_1$ for $i>1$ (i.e., $a_i=f(c_i,c_1)$). Note that the a_i's are RCF indiscernibles. Similarly, $b_i=f(c_i,c_2)$ for $i>2$ are also indiscernibles. Now for $i>2$ we have $a_i c_1=c_i=b_i c_2=b_i a_2 c_1$. So $b_i = \frac{a_i}{a_2}$. But by indiscernibility the type of the b_i's is the type of the a_i's. So it is a necessary condition that $c_i = \frac{e_i}{e_1} c_1$ for some set of RCF indiscernibles $\{e_i\}$. Now for existence, let a_i's be any type of RCF indiscernibles and let $c_i = \frac{a_i}{a_1} c_1$. Then

$c_k = \frac{a_k}{a_j} c_j$ for $k>j$. Since the type of the $\frac{a_k}{a_j}$'s is the same as the type of $\frac{a_i}{a_1}$'s we have that the c_i's are indiscernibles. □

<u>Proposition 3.4</u>: The RCF indiscernible types with $x_2>x_1$ and with the $\frac{x_i}{x_1}$'s the same type as the x_i's are:

A) $x_1>Q$ (i.e. $s=+\infty$)

1. x_i's are type I

2. x_i's are type IIB

B) $1<x_1<1+\varepsilon$ for $\varepsilon\epsilon Q^+$ (i.e. $s=1$)

1. $2(x_2-x_1)<(x_3-x_1)$

a) $\frac{1}{x_i-1}$ are type I

b) $\frac{1}{x_i-1}$ are type IIA

c) $\frac{1}{x_i-1}$ are type IIB

2. $2(x_2-x_1)>(x_3-x_1)$

a) $\frac{1}{x_\omega-x_i}$ are type I

b) $\frac{1}{x_\omega-x_i}$ are type IIA

c) $\frac{1}{x_\omega - x_i}$ are type IIB.

Proof: $x_2 > x_1 \rightarrow \frac{x_2}{x_1} > 1$. Hence $x_i > 1$ by hypothesis. If $x_i > 1+\varepsilon$ then the type is infinite and we are in Case A, otherwise we are in Case B. If x_i's are in Case A then $\frac{x_i}{x_1}$'s are also infinite and hence the x_i's are I, IIA or IIB. IIA is not possible because then $\frac{x_i}{x_1}$ would be type I. If the type is of Case B then clearly we are in subcases (1) or (2). Suppose we are in Case B1. Then $2(\frac{x_2}{x_1} - 1) < (\frac{x_3}{x_1} - 1)$. Hence the $\frac{1}{\frac{x_i}{x_1} - 1}$ are I, IIA or IIB. Hence $\frac{1}{x_i - 1}$ are I, IIA or IIB. This is enough to determine the type. Suppose the type is a B2. Then $\frac{1}{\frac{x_\omega}{x_i} - 1}$ are I, IIA or IIB.

Now $P(x_1, \ldots x_n) > 0 \leftrightarrow P(\frac{x_1}{x_o}, \ldots \frac{x_n}{x_o}) > 0 \leftrightarrow P(\frac{x_1}{x_\omega} \cdot \frac{x_\omega}{x_o}, \ldots \frac{x_n}{x_\omega} \cdot \frac{x_\omega}{x_o}) > 0$.

Since the type of $\frac{x_\omega}{x_1}$ is fixed, the polynomial is determined. Hence the type of the x_i's is determined. So we have uniqueness. For existence let e_i's be I, IIB, III(1)A1, III(1)A2, III(1)A3, III(1)B1, III(1)B2, or III(1)B3. Let $c_i = \frac{e_i}{e_1} v_o$ for soem fixed $v_o \in V$. One can routinely check that they meet the conditions in the outline. □

Corollary 3.5: The indiscernible types of a real vector space of dimension 1 with $x_i \in V$ are characterized by specifying which of the eight types of RCF indiscernibles from Proposition 3.4 the $f(x_i, x_1)$'s are.

Proof: Follows immediately by Proposition 3.3 and by Proposition 3.4. □

Note that all of these new types are interdefinable with type I RCF indiscernibles. So, as expected, we get no added structural information.

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