Random algebraic constructions

Boris Bukh

26 March 2022

Based on joint works with
Pavle Blagojević, David Conlon, Zilin Jiang,
Roman Karasev, Michael Tait

and on prior works
of many others

What is this talk?

- Random constructions of combinatorial objects
- Specific technique to correlate good events

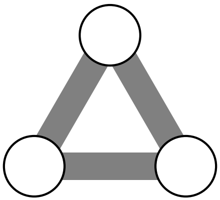
Motivational problem: Turán numbers

Forbidden subgraph F. How to make large F-free graph?

$$\mathrm{ex}(n, F) = \max_{\substack{G \text{ is } F\text{-free} \\ n \text{ vertices}}} e(G)$$

Erdős–Stone'46

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right) \binom{n}{2}$$



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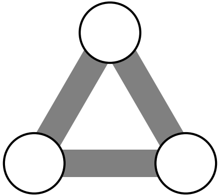


Useless for bipartite F

Turán numbers: complete bipartite case

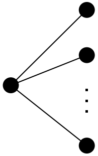
Theorem (Kovari–Sós–Turán)

The maximum number of edges in a $K_{s,t}$-free graph is $\mathrm{ex}(n, K_{s,t}) \le c_{s,t} n^{2-1/s}$

"Proof":

1. Pretend that the $K_{s,t}$-free graph is regular.
   Let d be the degree of each vertex.

2. Count s-stars



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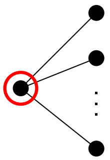

Exactly $n\binom{d}{s}$ copies

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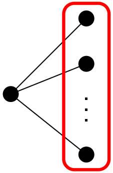

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At most $(t-1)\binom{n}{s}$ copies

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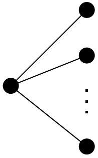


Exactly $n\binom{d}{s}$ copies

At most $(t-1)\binom{n}{s}$ copies

$\implies n\binom{d}{s} \le (t-1)\binom{n}{s}$

$\implies nd^s \lesssim n^s$

In a real proof, replace 1 by Jensen's inequality.

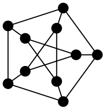
Turán numbers: complete bipartite case

Upper bound:

$$\text{ex}(n, K_{s,t}) \leq c_{s,t} n^{2-1/s}$$

Lower bound ideas:

Doodle



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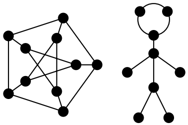
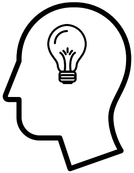


Be clever



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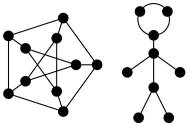

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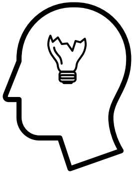
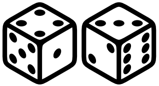


Try luck

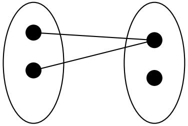


Naive construction: what does not work

Random graph with $n^{2-1/s}$ edges:

Construction:

- Bipartite graph on $n + n$ vertices
- Edge probability is $p = n^{-1/s}$



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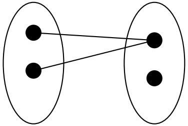

Analysis:

- Fix any s vertices on the left
- N are their common neighbors
- $\Pr[y \in N] = p^s = 1/n$
- $\mathbb{E}[|N|] = 1$. Good news!

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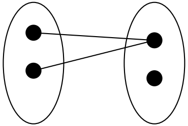

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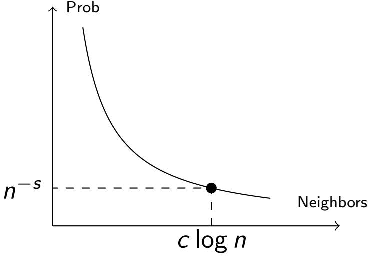
- There are $\binom{n}{s} \approx n^s$ sets.
- For each, $N \sim$ Poisson. Bad news!



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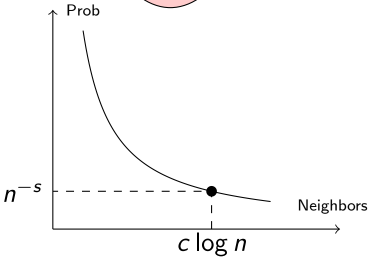
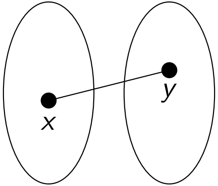
$K_{s,t}$-free with
$t \approx \log n$



Motivation: Being clever

The maximum number of edges in a $K_{2,2}$-free graph is

$$\mathrm{ex}(n, K_{2,2}) = \Theta(n^{3/2}).$$



$\mathbb{F}_q^2$ $\mathbb{F}_q^2$

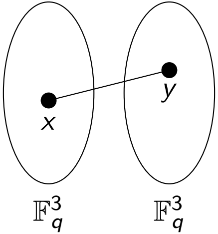
Connect $x = (x_1, x_2)$ with $y = (y_1, y_2)$ if $x_1y_1 + x_2y_2 = 1$.

$2q^2$ vertices
degree q

Motivation: Being clever

The maximum number of edges in a $K_{3,3}$-free graph is

$$\mathrm{ex}(n, K_{3,3}) = \Theta(n^{2-1/3}).$$



Connect $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ if $(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 = 1$.

$2q^3$ vertices

degree $\approx q^2$

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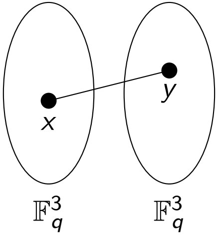


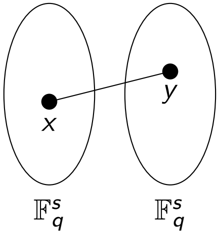
Connect $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ if $(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 = 1$.

No similar $K_{4,4}$-free graph

$2q^3$ vertices
degree $\approx q^2$

More complicated $K_{s,t}$-free graph with $t > (s-1)!$

Try luck: random algebraic construction



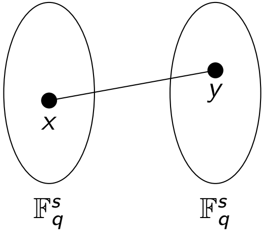
Connect $x = (x_1, \ldots, x_s)$ and $y = (y_1, \ldots, y_s)$ if $f(x, y) = 0$.

Choose f randomly among all polynomials of degree d.

Good news 1: Behaves randomly on small scale.

Good news 2: Very correlated on large scale.

Small-scale independence



$x \sim y$ if $f(x, y) = 0$

Random f of deg d

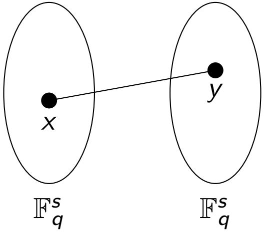
Claim

For any $x_1, \ldots, x_a \in \mathbb{F}_q^s$ and $y_1, \ldots, y_b \in \mathbb{F}_q^s$, the edges $(x_i y_j : i, j)$ are independent, if $d \geq d_0(a, b)$.

Intuition:

- *Every* function is a polynomial of degree $q - 1$.
- Claim holds for a random function

Small-scale independence



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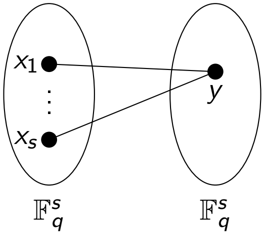
Key proof steps:

- Unique degree-d polynomial through $d + 1$ pts
- Generic rotation of the coordinates

Large-scale correlation



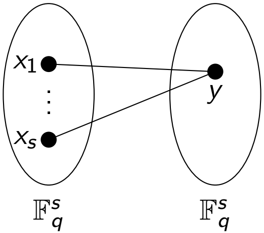
Common neighborhood of $A = \{x_1, \ldots, x_s\}$ is

$$N(A) = \{y \in \mathbb{F}_q^s : f(x_1, y) = \cdots = f(x_s, y) = 0\}$$

Analogies:

Linear equations　　　Polynomial equations

Large-scale correlation



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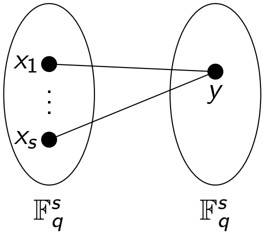
Analogies:

Linear equations	Polynomial equations
Subspace	Variety

Large-scale correlation



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Dimension d	"Dimension" d

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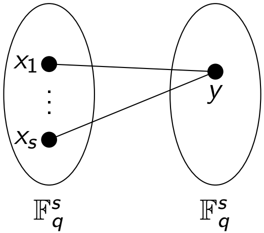

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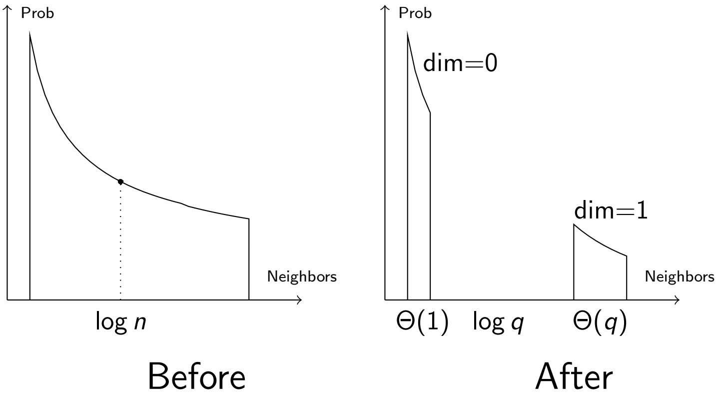
Analogies:

Linear equations	Polynomial equations
Subspace	Variety
Dimension d	"Dimension" d
q^d points	$\Theta(q^d)$ points

Neighborhood size: punchline

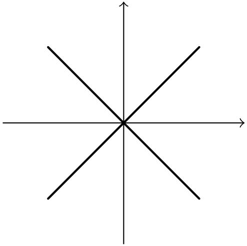


Let $t \gg 1$,

$$
\begin{aligned}
\Pr[\exists K_{s,t} \text{ subgraph}] &= \Pr[\exists A \text{ s.t. } |N(A)| \geq t] \\
&= \Pr[\exists A \text{ s.t. } |N(A)| \geq \Theta(q)] \\
&= \text{tiny}
\end{aligned}
$$

Dimension and "dimension"

$x^2 - y^2 = 0$

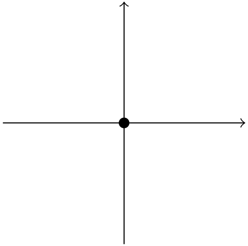


Dimension is 1

$2q$ pts in $\mathbb{F}_q^2$

"Dimension" is 1

$x^2 + y^2 = 0$



Dimension is 1

1 pt in $\mathbb{F}_q^2$

if $q \equiv 3 \pmod 4$

"Dimension" is 0

Dimension and "dimension"

Dimension is well-behaved over $\overline{\mathbb{F}_q}$ (algebraically closed)

For variety V, irreducible decomposition $V = V_1 \cup \cdots \cup V_k$.

Examples:

1. $\{x^2 - y^2 = 0\}$ is $\{x - y = 0\} \cup \{x + y = 0\}$
2. $\{x^2 + y^2 = 0\}$ is $\{x + iy = 0\} \cup \{x - iy = 0\}$

Theorem (Lang–Weil)

If variety V is irreducible over $\overline{\mathbb{F}_q}$, then the number of points V over $\mathbb{F}_q$ is $q^{\dim V}(1 + o(1))$.

Problem:

What if V is irreducible over $\mathbb{F}_q$ but not over $\overline{\mathbb{F}_q}$?

Dimension and "dimension"

Problem

What if V is irreducible over $\mathbb{F}_q$ but not over $\overline{\mathbb{F}_q}$?

Irreducible decomposition

$$V = V_1 \cup \cdots \cup V_k$$

Map $\mathrm{Frob}\colon x \mapsto x^q$ generates $\mathrm{Gal}(F/\mathbb{F}_q)$ for every extension $F/\mathbb{F}_q$.

Dimension and "dimension"

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Map Frob: $x \mapsto x^q$ generates $\mathrm{Gal}(F/\mathbb{F}_q)$ for every extension $F/\mathbb{F}_q$.

Corollary

1. Frob *permutes* $V_1, \ldots, V_k$, *and does so transitively*
2. $V(\mathbb{F}_q) = V_i(\mathbb{F}_q)$

Proof:

1. If $V_1, \ldots, V_t$ is an orbit, then $V_1 \cup \cdots \cup V_t$ is an $\mathbb{F}_q$-component
2. The Frobenius map does not move $\mathbb{F}_q$-points

Dimension and "dimension"

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It follows that

$$V(\mathbb{F}_q) = V_1(\mathbb{F}_1) \cap \cdots \cap V_k(\mathbb{F}_q)$$

Induction on the dimension of V

Technicalities and an embellishment

The iteration:

1. Break the variety into $\mathbb{F}_q$-components
2. Break each $\mathbb{F}_q$-component V into $\overline{\mathbb{F}_q}$-components $V_1, \ldots, V_k$
3. Replace V by $V_1 \cap \cdots \cap V_k$
4. Repeat

Key technical problem

Must control the number of components

Technicalities and an embellishment

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Key technical problem

Must control the number of components

Solution 1

- Number of components is $\leq \deg V$
- $\deg(U \cap V) \leq (\deg V)(\deg U)$
  (Bezout's inequality)

Technicalities and an embellishment

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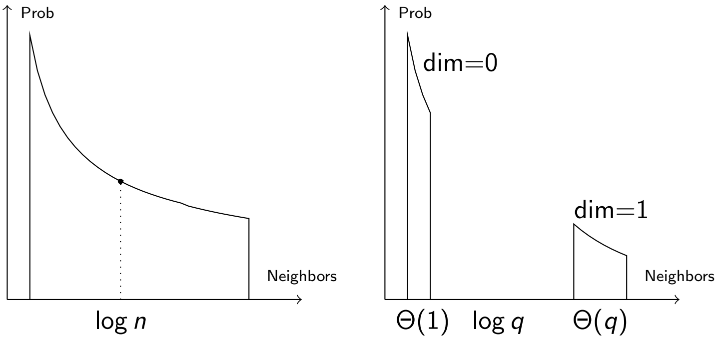
Solution 2

Get rid of probability

Do dimension-counting

Back to neighborhood size



Before

After

Let $t \gg 1$,

$$\begin{aligned} \Pr[\exists K_{s,t} \text{ subgraph}] &= \Pr[\exists A \text{ s.t. } |N(A)| \geq t] \\ &= \Pr[\exists A \text{ s.t. } |N(A)| \geq \Theta(q)] \\ &= \text{tiny} \end{aligned}$$

This step

Probabilistic argument

Goal

An upper bound on $\Pr\left[|N(A)| \geq T\right]$, for $T = \Theta(q)$

Known facts:

- If edges were independent, $N(A) \approx \text{Poisson}(1)$.
- Edges are k-wise independent, for large k.

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1. ℓ'th moment of $N(A)$

$$\mathbb{E}\left[N(A)^{\ell}\right] = \sum_{v_1,\ldots,v_\ell} \mathbb{E}\left[\mathbf{1}_{v_1 \in N(A)} \cdot \ldots \cdot \mathbf{1}_{v_e ll \in N(A)}\right]$$

is the same as that of Poisson(1)

2. By Markov's inequality

$$\Pr\left[N(A) \geq T\right] = \Pr\left[N(A)^{\ell} \geq T^{\ell}\right] \leq \frac{\mathbb{E}\left[N(A)^{\ell}\right]}{T^{\ell}} = \frac{O(1)}{T^{\ell}}$$

Dimension-counting argument

Goal

An upper bound on $\Pr\left[|N(A)| \geq T\right]$, for $T = \Theta(q)$

Key points:

1. $$\begin{aligned}\Pr\left[|N(A)| \geq \Omega(q)\right] &= \Pr\left[\text{“dimension” of } N(A) \geq 1\right] \\ &\leq \Pr\left[\text{ dimension of } N(A) \geq 1\right]\end{aligned}$$

2. Dimension-d variety behaves similarly to a set of size q^d.
   Example (pigeonhole principle):

$$V_x = \{y : (x, y) \in V\}$$
$$U = \{x : \dim V_x \geq d\}$$

then

$$\dim U \leq \dim V - d$$

Dimension-counting argument

Goal

An upper bound on $\Pr\big[|N(A)| \geq T\big]$, for $T = \Theta(q)$

Key points:

1. $$\begin{aligned}\Pr\big[|N(A)| \geq \Omega(q)\big] &= \Pr\big[\text{“dimension” of } N(A) \geq 1\big] \\ &\leq \Pr\big[\text{ dimension of } N(A) \geq 1\big]\end{aligned}$$

2. Dimension-d variety behaves similarly to a set of size q^d.
   Example (pigeonhole principle):

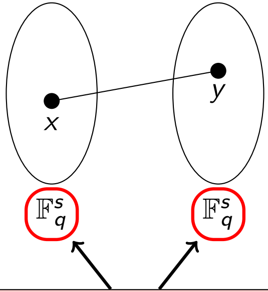
   $$V_x = \{y : (x, y) \in V\}$$
   $$U = \{x : \dim V_x \geq d\}$$

   then

   $$\dim U \leq \dim V - d$$

The rest is details

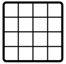
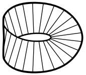
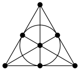
Improvement



What does $\mathbb{F}_q^s$ look like?

Improvement

$\mathbb{F}_q^s$ looks like…

a)  b)  c)  d)  e) 

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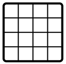
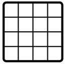
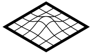


(flat)

Improvement

$\mathbb{F}_q^s$ looks like. . .  (flat)

... but it better be like  (wobbly)

Theorem (B.)

The exist $K_{s,t}$-free graphs with $c_s n^{2-1/s}$ edges and $t \leq C^s$.

Another problem

Complete bipartite graphs:

$\mathrm{ex}(n, K_{s,t}) \sim n^{2-1/t}$ if $s \gg t$

Cycles:

$\mathrm{ex}(n, C_{2\ell}) \leq c_\ell n^{1+1/\ell}$ sharp for $\ell = 2, 3, 5$

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"Proof":

1. Pretend that the C_{2t}-free graph is d-regular.
2. Pretend that the graph is in fact $\{C_3, C_4, \ldots, C_{2\ell}\}$-free

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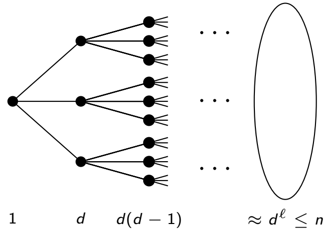
3. 

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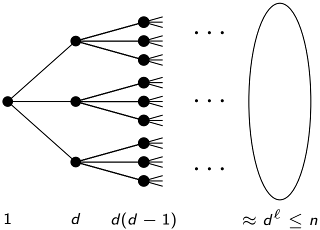
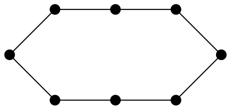
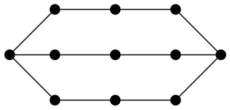
3. 

Actual proofs for cycles **are messy**

Theta graphs



Theta graph $\Theta_{4,2} = C_8$



Theta graph $\Theta_{4,3}$

Upper bound (Faudree–Simonovits):

$\mathrm{ex}(n, \Theta_{\ell,t}) \leq c_{\ell,t} n^{1+1/\ell}$

Lower bound (Conlon):

$\mathrm{ex}(n, \Theta_{\ell,t}) \geq \frac{1}{2} n^{1+1/\ell}$ for $t \geq t(\ell)$

Conlon's construction

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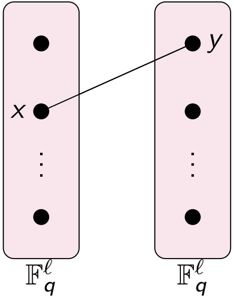

Edges:

$x \sim y$ if

$f_1(x, y) = \cdots = f_{\ell-1}(x, y) = 0$

Random $f_1, \ldots, f_\ell$

Average degree is $nq^{-(\ell-1)} = q$

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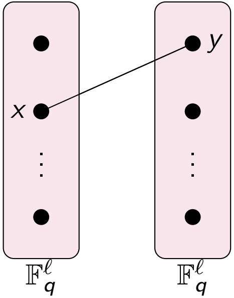

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Path $x_1 y_1 x_2 y_2 \cdots$ is a solution to $f(x_1, y_1) = f(x_2, y_2) = \cdots = 0$

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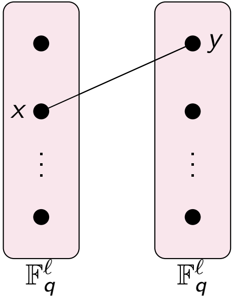

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Key point: If U, V are varieties, then $U \setminus V$ has "dimension"

Theta graphs, more carefully

Upper bound (B.–Tait):

For any ℓ, we have $\mathrm{ex}(n, \Theta_{\ell,t}) \leq c_\ell t^{1-1/\ell} \cdot n^{1+1/\ell}$

Lower bound (B.–Tait):

For odd ℓ, we have $\mathrm{ex}(n, \Theta_{\ell,t}) \geq c'_\ell t^{1-1/\ell} \cdot n^{1+1/\ell}$

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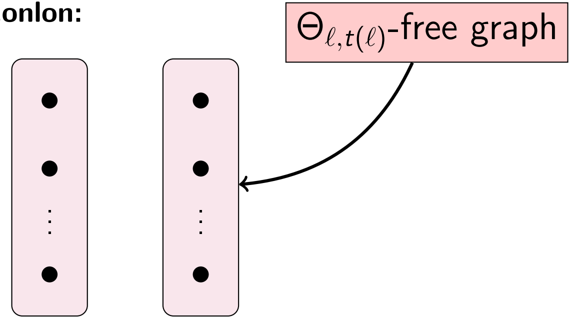

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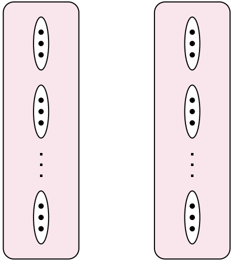

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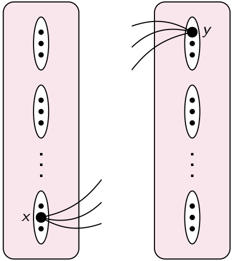

Consider $\Theta_{\ell,T}$:

Endpoints x, y

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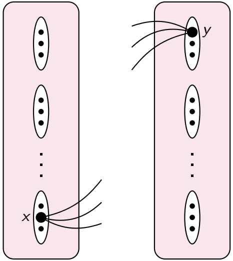


Consider $\Theta_{\ell,T}$:

Endpoints x, y

Key observation:

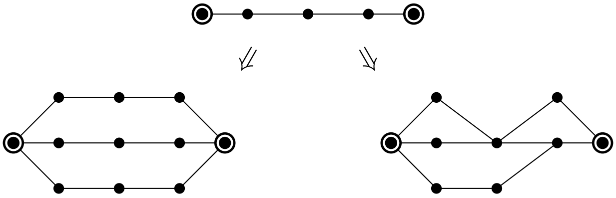
x, y are in different blobs because ℓ is odd

Conclusion:

$\Theta_{\ell,T/c}$ in original

Turán exponents

Clonining the path thrice in two waysx:



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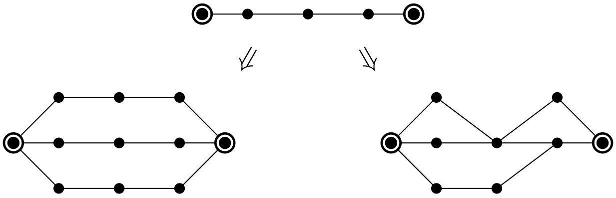

Generally:

T is a rooted tree (with several roots)

$\mathcal{T}^p$ consists of all p-fold clones of T

Theorem (B.–Conlon)

For every rational number $r \in [1, 2]$, there is a T such that

$$\mathrm{ex}(n, \mathcal{T}^p) = \Theta(n^r)$$

for all $p \geq p_0$.

References

Upper bounds for cycles and thetas:
Kovari–Sós–Turán, Bondy–Simonovits, Faudree–Simonovits, Verstraëte, Pikhurko, B.–Jiang, B.–Tait

$K_{s,t}$-free constructions:
Erdős–Rényi–Sós, Brown, Füredi, Kollár–Rónyai–Szabó, Alon–Rónyai–Szabó, B.–Blagojević–Karasev, Ball–Pepe, B.

Theta-free constructions:
Brown, Benson, Wenger, Füredi, Lazebnik–Ustimenko–Woldar, Conlon, Verstraëte–Williford, B.–Tait

Thanks to all!