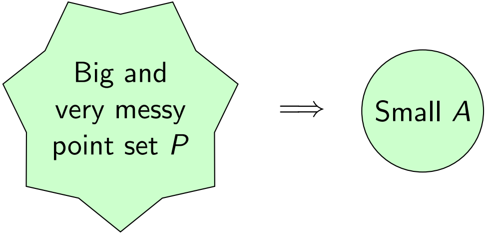
One-sided epsilon-approximants

Boris Bukh

May 2017

Joint with Gabriel Nivasch

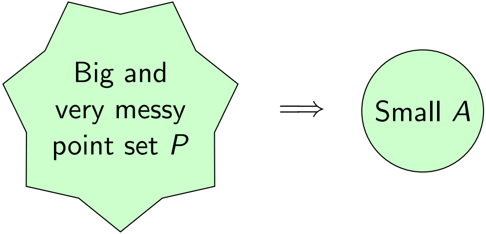
ε-approximants



Set $A \subset \mathbb{R}^2$ is an ε-approximant for set $P \subset \mathbb{R}^2$ w.r.t. to triangles if

$$\forall \text{ triangle } T \quad \left| \frac{|T \cap P|}{|P|} - \frac{|T \cap A|}{|A|} \right| \leq \varepsilon$$

ε-approximants



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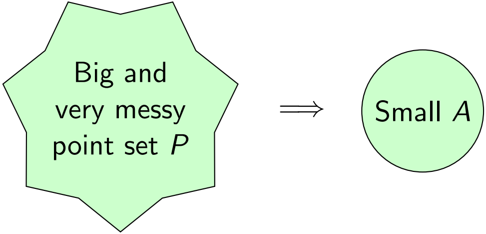
Theorem (VC'71)

For every $\varepsilon > 0$ and every $P \subset \mathbb{R}^2$ there is a ε-approximant w.r.t. triangles of size $f(\varepsilon)$.

ε-approximants



Set $A \subset \mathbb{R}^2$ is an ε-approximant for set $P \subset \mathbb{R}^2$ w.r.t. to thingies if

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For 'simple' shapes

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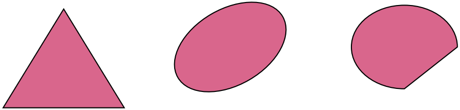
Simple shapes

Examples of simple shapes:



[Technical condition: bounded VC dimension.]

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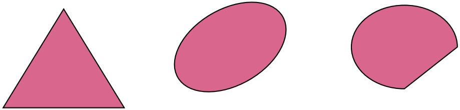

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Convex sets are not simple!

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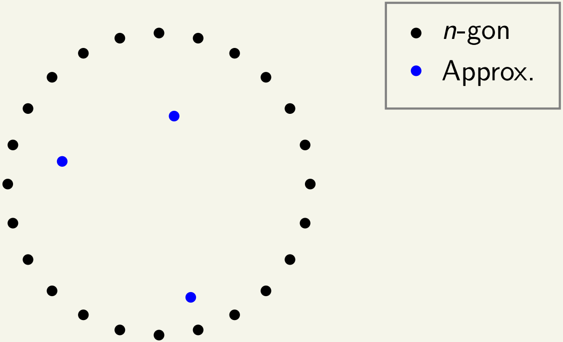
Theorem

There is no ε-approximant of size $f(\varepsilon)$ w.r.t. convex sets.

Proof.



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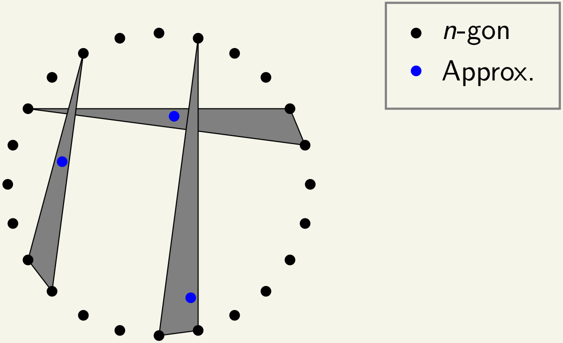

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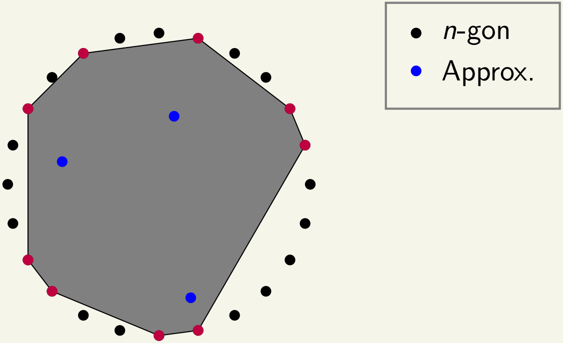

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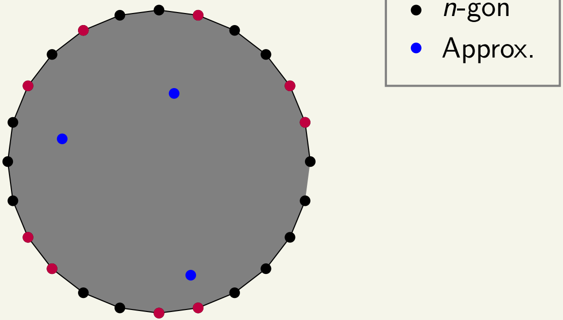
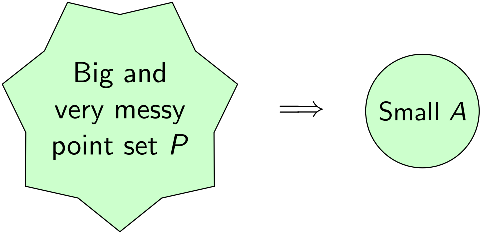


Simplification due to Dömötör Pálvölgyi

□

ε-nets



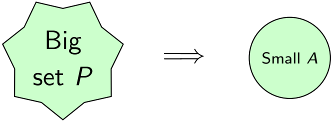
Set $N \subset \mathbb{R}^2$ is an ε-net for set $P \subset \mathbb{R}^2$ w.r.t. to family $\mathcal{F}$ if

$$\forall\, T \in \mathcal{F} \quad \frac{|T \cap P|}{|P|} > \varepsilon \implies T \cap N \neq \emptyset$$

Theorem (ABFK'92)

For every P there is an ε-net w.r.t. convex sets of size $f(\varepsilon)$.

One-sided ε-approximants



Set $A \subset \mathbb{R}^2$ is a one-sided ε-approximant for set $P \subset \mathbb{R}^2$ (w.r.t. to convex sets) if

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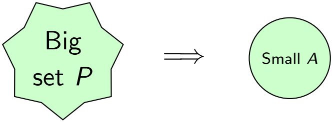

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ε-approximant $\Longrightarrow$ one-sided ε-approximant $\Longrightarrow$ ε-net

Theorem (B.–Nivasch)

For every P there is a one-sided ε-approximant of size $f(\varepsilon)$.

Proof outline

Theorem (B.–Nivasch)

For every P there is a one-sided ε-approximant of size $f(\varepsilon)$.

1. Replace P by a bounded-sized $\hat{P}$
2. Partition $\hat{P}$ into large subsets $S_1, S_2, \ldots$ in convex position
3. Explicit construction for sets in convex position

Proof outline

Theorem (B.–Nivasch)

For every P there is a one-sided ε-approximant of size $f(\varepsilon)$.

1. Replace P by a bounded-sized $\hat{P}$
   How: regularity lemma for semialgebraic relations
   Price: one-sided approx. for $\hat{P}$ must be 'semi-algebraic'

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Theorem (FGLNP'12,FPS'15)

Every $P \subset \mathbb{R}^2$ can be equipartitioned into few mostly-regular parts.

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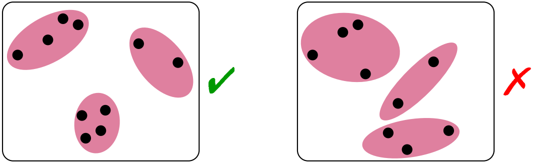
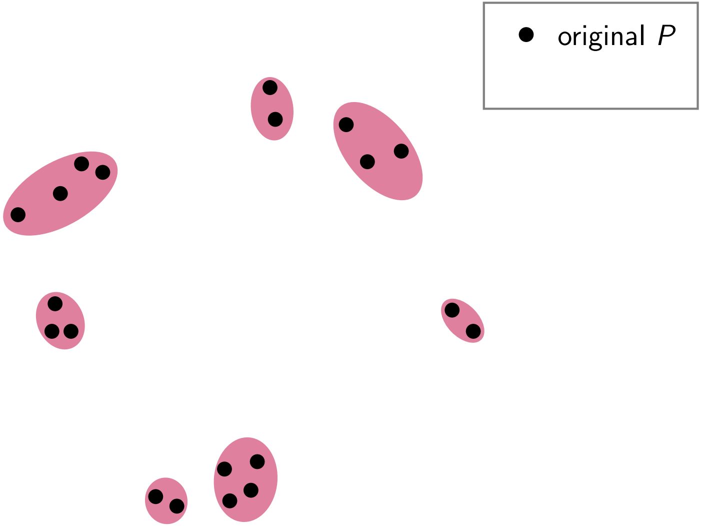


Mostly-regular = at most ε fraction of the triples are irregular

'Semi-algebraicity'



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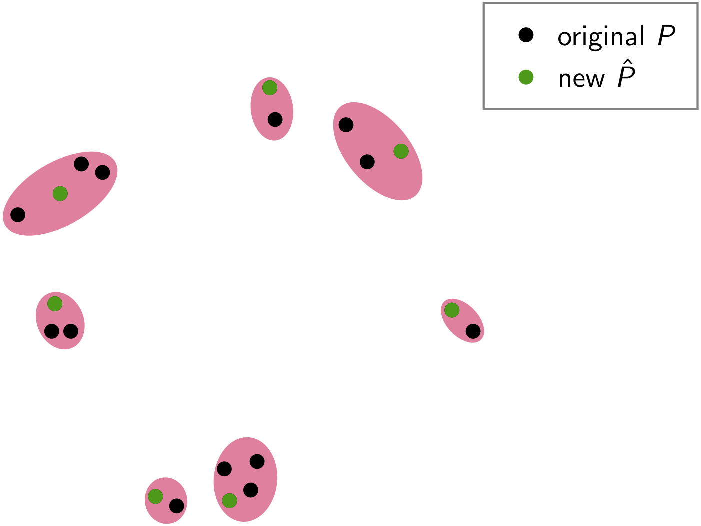

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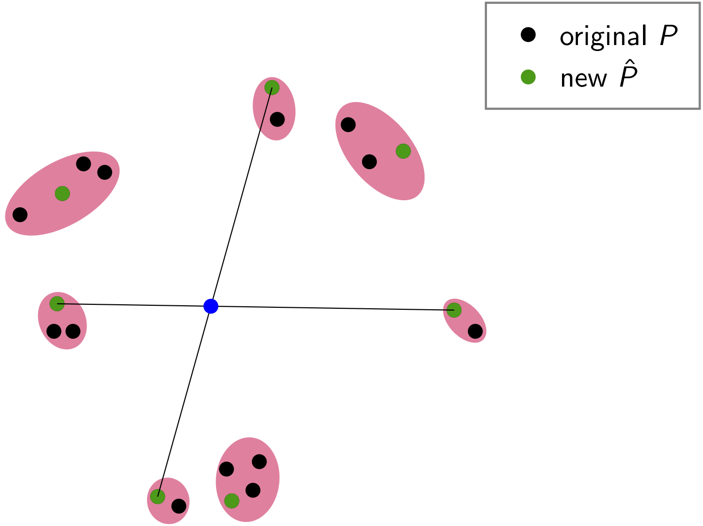
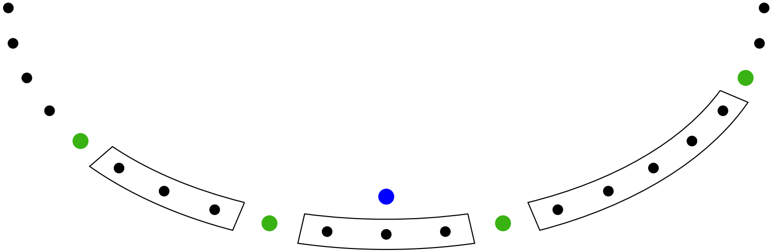


Point selection



Metaphorical drawing!

Proof outline: step 2

2. Partition $\hat{P}$ into large subsets $S_1, S_2, \ldots$ in convex position
   How: greedily via Erdős–Szekeres
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Theorem (ES'35)

Every P of size m contains a convex subset of size $\frac{1}{2}\log m$.

Corollary

Every P of size m can be partitioned into convex subsets of size $\frac{1}{4}\log m$ and a small leftover.

Proof outline: Step 3

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   How: regularity lemma for words
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Betting scenario:

Bet:	😐	😐	⑤	😐	⑤
Outcome:	?	?	?	?	?

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Coin is rigged!

Betting against a rigged coin

Coin flips:

? ? ? ? ? ? ? ? ? ? ? ? ? ? ? ?

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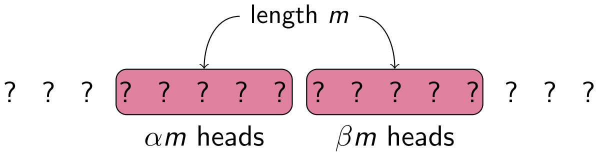

Approximation bet:

$\lvert\alpha - \beta\rvert \leq \varepsilon$	Win \$1
$\lvert\alpha - \beta\rvert > \varepsilon$	Lose \$1

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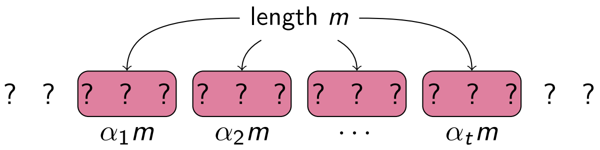

Approximation bet (t intervals):

$$\forall i, j \, |\alpha_i - \alpha_j| \leq \varepsilon \quad \text{Win \$1}$$
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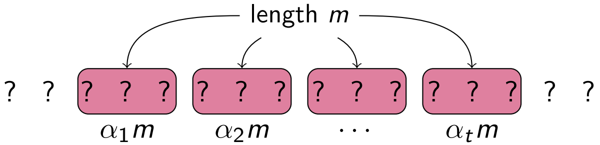

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Theorem (APP'13,FKT'14)

For every $\varepsilon > 0$ and every t, one can win against a rigged coin!

Rigged coin: strategy

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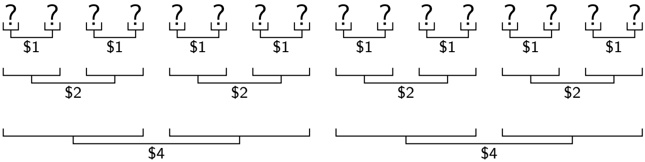
Strategy (t=2):



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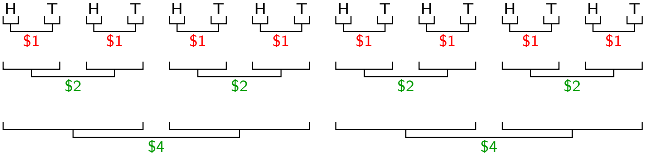

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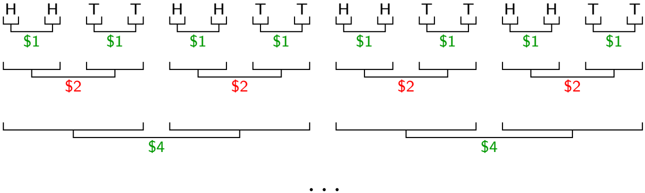

It is over!