Sharp bounds on the finite field Kakeya problem

or, Taming hedgehogs with polynomials

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June 16, 2022

Joint w/Ting-Wei Chao

16/Jun/2022

Kakeya problem in $\mathbb{R}^n$

Kakeya 1917

Find a figure of the least area on which a segment of length 1 can be turned 360° by a continuous movement.

Besicovitch 1928

There exist such sets of arbitrarily small area.

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Kakeya set in $\mathbb{R}^n$

A set containing a line segment in every direction

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Conjecture:

dim(Kakeya set) $= n$?

Best known:

dim(Kakeya set) $> cn$ for some $c < 1$

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Finite fields

Line

A set of the form $\{a + bt : t \in \mathbb{F}_q\}$

Kakeya set

A set containing a line in every direction

m-dimensional

Set of about q^m points

Wolff 1999

Does every Kakeya set in $\mathbb{F}_q^n$ contain $c_n q^n$ points?

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First line q points

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⋮

$q+(q-1)+\cdots+1=\binom{q+1}{2}$

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Bounds on Kakeya sets

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Does every Kakeya set in $\mathbb{F}_q^n$ contain $c_n q^n$ points?

Dvir 2009

At least $\Omega(q^{n-1})$ points.

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Yes, with $c_n = 1/n!$.

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Thm. (B.–Chao)

Yes, with $c_n = 2^{n-1}$.

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Dvir's argument

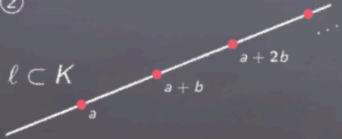
Every Kakeya set $K \subseteq \mathbb{F}_q^n$ has at least $\binom{q+n-1}{n}$ elements.

Proof

① Find polynomial f vanishing on K

- $f(p) = 0$ is a linear equation on f
- dim(Degree-d polynomials) $= \binom{d+n}{n}$
- Take $d = q - 1$

②



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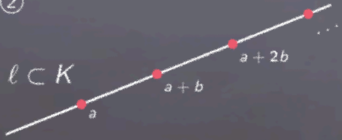

- ► $\ell = \{a + bt : t \in \mathbb{F}_q\}$
- ► $f(a + bt) = 0$ on $\mathbb{F}_q$
- ► $f(a + bt)$ is zero poly.
- ► $[t^{\deg f}]f(a + bt) = 0$

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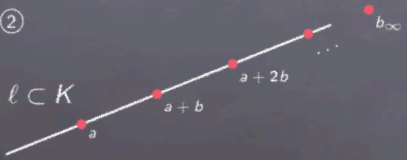

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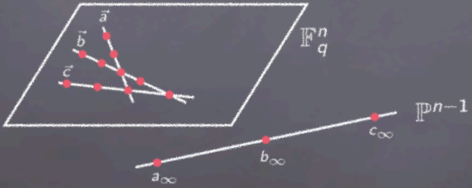

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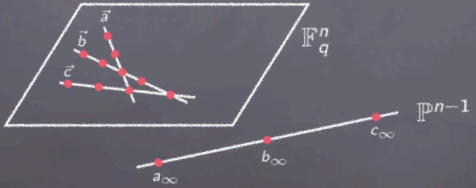

④ $f = 0$ on $\mathbb{P}^{n-1}$

⑤ $f_d = 0$

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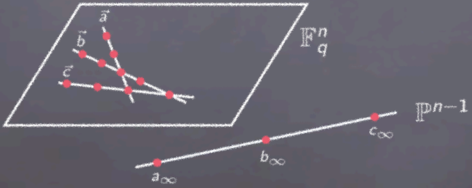

④ $f = 0$ on $\mathbb{P}^{n-1}$ $\longleftarrow$ q^{n-1} conditions

⑤ $f_d = 0$ $\longleftarrow$ $\binom{q+n-1}{n-1}$ coefficients

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Our argument for $n = 3$

Every Kakeya set $K \subseteq \mathbb{F}_q^3$ has at least $\frac{1}{4}(q^3 + q^2)$ elements.

Proof

① Better space of polynomials

$$A \stackrel{\text{def}}{=} \{(\alpha_1, \alpha_2, \alpha_3) : \alpha_1 + \alpha_2 + \alpha_3 < 2q \text{ and } \alpha_1, \alpha_2 < q\}$$

$$V \stackrel{\text{def}}{=} \left\{ \sum_{\alpha \in A} c_\alpha x^\alpha : c_\alpha \in \mathbb{F}_q \right\}.$$

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$$\begin{cases} f(p) = 0 \\ \nabla f(p) = 0 \end{cases} \quad \text{for every } p \in K.$$

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$\dim V = q^3 + q^2$

② Different vanishing condition

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$$\begin{cases} f(p) = 0 \\ \nabla f(p) = 0 \end{cases}$$ for every $p \in K$.

4 conditions per point

$\Downarrow$

$4|K| \geq q^3 + q^2$

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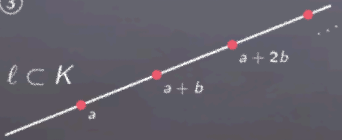

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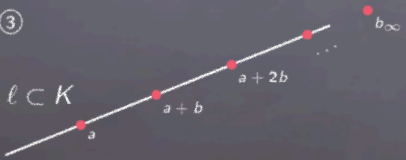

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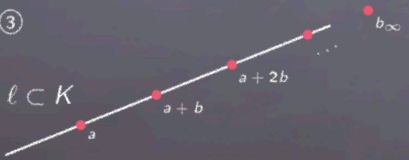

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④ $g(x, y) = f_d(x, y, 1)$ vanishes

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Our argument, general

- The $n = 3$ does NOT generalize
- Vanishing order to $\to \infty$ — DKSS 2013
- The conditions depend on p and ℓ — Zhang 2020

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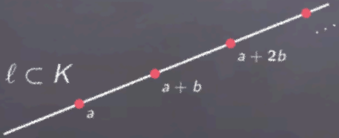

- $\ell = \{a + bt : t \in \mathbb{F}_q\}$
- $g = D^\alpha f$ with $|\alpha| < r$
- $\operatorname{ord} g(a + bt) \geq s$ everywhere on $\mathbb{F}_q$

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No idea why it is sharp!

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Lower-order terms

Thm. (B.–Chao)

Every Kakeya set in $\mathbb{F}_q^n$ satisfies $|K| \geq 2^{-n+1} q^n (1 + \frac{n-1}{2q})$.

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