Stabbing simplices by points and affine spaces

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Joint work with Jiří Matoušek and Gabriel Nivasch.

Introduction

The setting

Let S be a set of n points in $\mathbb{R}^d$. Every $d+1$ of them spans a simplex, for the total of $\binom{n}{d+1}$ simplices. Find a point that stabs (lies in) as many simplices as possible.

Theorem (Bárány'82)

There is always a point stabbing a positive fraction $c_d > 0$ of all the simplices.

Problem

How large is this positive proportion c_d?

Lower bounds

Theorem (Boros-Füredi'84, B.'06)

In the plane there is a point stabbing $n^3/27$ triangles (a $2/9$ fraction of the triangles).

Theorem (Bárány'82)

There is a point stabbing

$$\frac{1}{d!}\left(\frac{n}{d+1}\right)^{d+1}$$

d-simplices.

Lower bounds

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In the plane there is a point stabbing $n^3/27$ triangles (a $2/9$ fraction of the triangles).

Theorem (Wagner'03)

There is a point stabbing

$$\frac{d^2+1}{d+1}\frac{1}{d!}\left(\frac{n}{d+1}\right)^{d+1}$$

d-simplices.

Upper bounds

Theorem (Bárány'82)

If S is any set in general position in $\mathbb{R}^d$, then no point stabs more than

$$2^{-d}\binom{n}{d+1}$$

d-simplices.

Theorem (Boros-Füredi'84)

There is a planar point set, such no point stabs more than $(1/27 + 1/729)n^3$ triangles.

They claimed the bound $1/27$ (which would be sharp), but the proof is wrong.

New upper bounds

Theorem (B.-Matoušek-Nivasch)

There is a set S of n-points such that no points stabs more than

$$\left(\frac{n}{d+1}\right)^{d+1}$$

d-simplices.

- Sharp for $d = 2$.
- Very simple construction.

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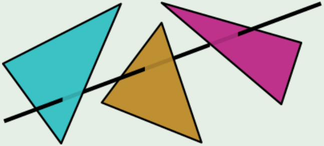
Generalization

General problem

Let S be a set of n points in $\mathbb{R}^d$. Every $d-k+1$ of them spans a $(d-k)$-simplex, for the total of $\binom{n}{d-k+1}$ simplices. Find a k-dimensional affine subspace (k-flat) that stabs as many simplices as possible.

Example

If $d = 3$ and $k = 1$, then we want to stab as many triangles in $\mathbb{R}^3$ as possible with a line.



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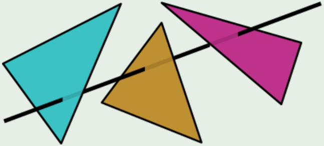

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Let S be a set of n points in $\mathbb{R}^d$. Every $d - k + 1$ of them spans a $(d - k)$-simplex, for the total of $\binom{n}{d-k+1}$ simplices. Find a k-dimensional affine subspace (k-flat) that stabs as many simplices as possible.

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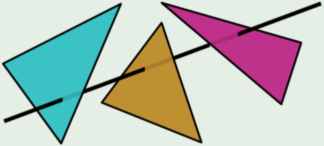

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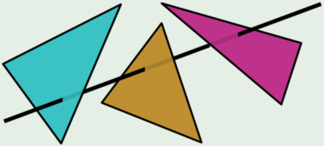

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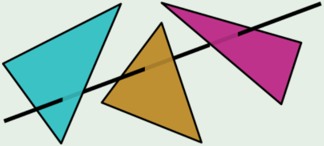

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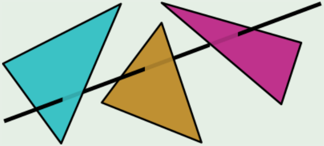

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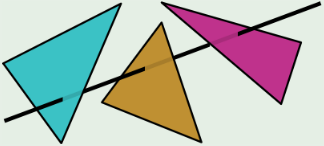

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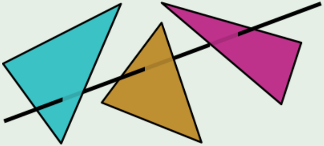

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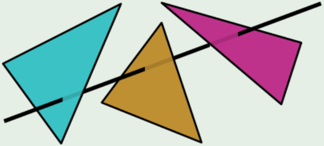

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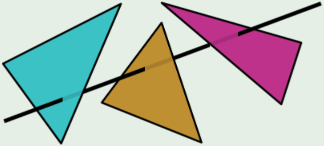

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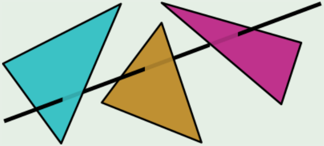

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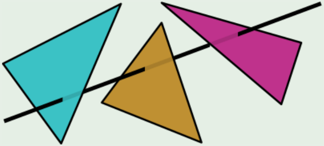

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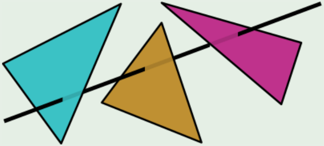

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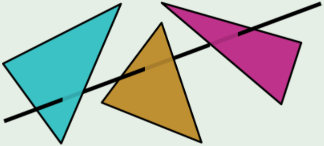

Simple idea

Project the point into xy plane, and apply Boros-Füredi result. We obtain a line stabbing $n^3/27$ triangles.

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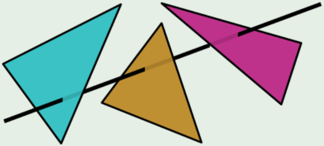

Naïve idea

Project the point into *xy* plane, and apply Boros-Füredi result. We obtain a line stabbing $n^3/27$ triangles.

Theorem (B.-Matoušek-Nivasch)

There is a line stabbing $n^3/25$ triangles.

Why care?

Trivial

Stabbing $n^3/27$ triangles.

Non-trivial

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Answer #1

Recall: Bárány('82) showed that for no point set $S \subset \mathbb{R}^2$ there is a point stabbing more than $n^3/24$ triangles. Hence for no point set $S \subset \mathbb{R}^3$ there is a line stabbing more than $n^3/24$ triangles.

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Answer #2

For the analogous problem of stabbing sparse families of triangles the gap might be worse.

A different problem

Let S be a set of n points in $\mathbb{R}^d$. Every 3 of them spans a triangle, for the total of $\binom{n}{3}$ triangles. **Let T be a family of any m of these triangles**. Find a $d-2$-flat that stabs as many triangles **in T** as possible.

Eppstein'93

For the triangles in the plane ($d = 2$) there is a point that stabs

$$\frac{m^3}{n^6 \log^2 n}$$

triangles.

Dey-Edelsbrunner'94, Smorodinsky'03

For the triangles in the space ($d = 3$) there is a line that stabs

$$\frac{m^3}{n^6}$$

triangles.