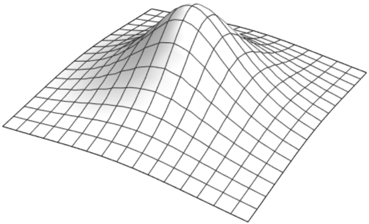
Extremal graphs without exponentially-small bicliques

Boris Bukh

April 2023



Turán numbers

Paul Turán 1941:

$$\mathrm{ex}(n, F) = \max e(G) \text{ in an } F\text{-free} \atop \text{n-vertex graph } G$$

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$$\text{n-vertex graph } G$$

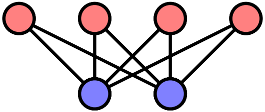
Short summary:

$$\mathrm{ex}(n, F) = \text{asymptotic formula} \qquad \text{if } \chi(F) > 2,$$
$$\mathrm{ex}(n, F) = \text{mystery} \qquad \text{if } \chi(F) = 2.$$

Turán numbers: bipartite case

Kővári–Sós–Turán 1954:

$$\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/s})$$



$K_{2,4}$

Turán numbers: Bipartite case

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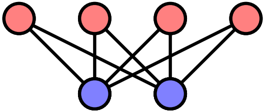

$K_{2,4}$

Sharp:

$s = 2$

$s = 3$

$t \gg s$

Turán numbers: Bicliques

$K_{s,t}$-free with $\Omega(n^{2-1/s})$ edges

$t > s!$	Kollár, Rónyai, Szabo 1996
$t > (s-1)!$	Alon, Rónyai, Szabo 1999

Theorem

There are extremal $K_{s,t}$-free graphs for

$$t > 9^{s+O(s^{2/3}\log s)}.$$

Turán numbers: Bicliques

$K_{s,t}$-free with $\Omega(n^{2-1/s})$ edges

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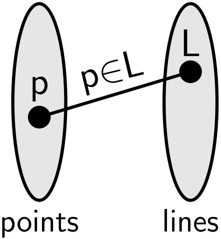
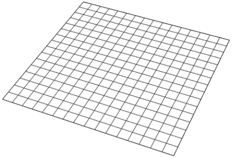
Bumpy
space

Turán numbers: bicliques

$K_{2,2}$-free





plane $\mathbb{F}_q^2$

Theorem

There are extremal $K_{s,t}$-free graphs

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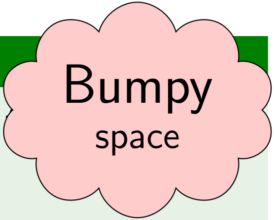
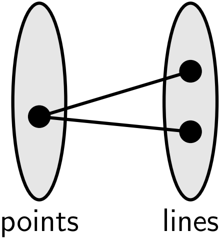
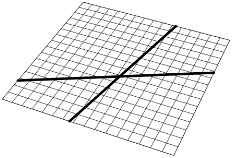


Turán numbers: Bicliques

$K_{2,2}$-free





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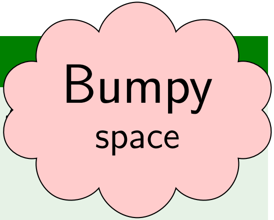
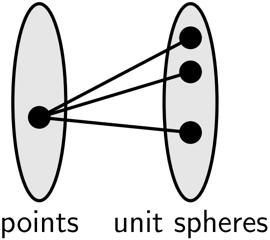
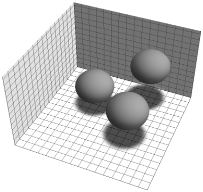


Turán numbers: bicliques

$K_{3,3}$-free





space $\mathbb{F}_q^3$

Theorem

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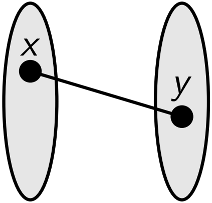
Bumpy
space

Bumpy?

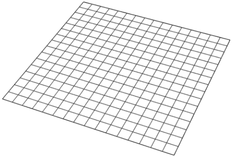
Graph



$x \sim y$

$f(x, y) = 0$

Vertex set



Bumpy?

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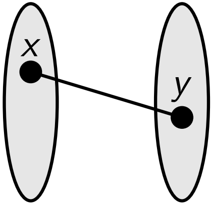

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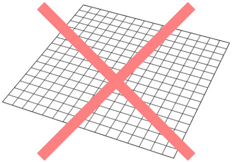

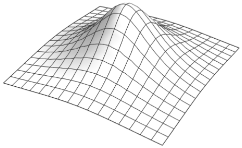

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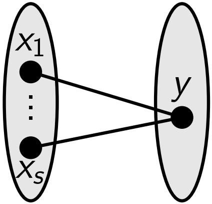


$$N(x_1, \ldots, x_s)$$

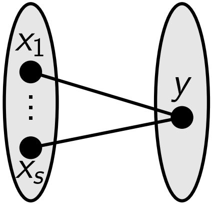
$$f(x_1, \cdot) = 0$$

$$\vdots$$

$$f(x_s, \cdot) = 0$$

Bumpy?

Graph



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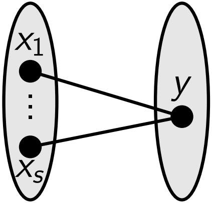
Bezout:

$$\deg f = d$$
$$\Downarrow$$
$$d^s \text{ solns.}$$

(maybe)

Bumpy?

Graph



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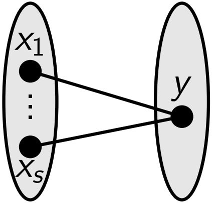
Low-degree f?

Bumpy?

Graph



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Trouble:

$\deg f = d$

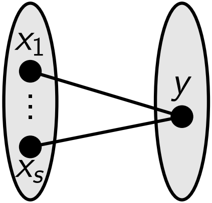
collinear $x_1, \ldots, x_s$

$$x_1 \sim y, \ldots, x_{d+1} \sim y$$
$$\Downarrow$$
$$\text{all } x_i \sim y$$

Low-degree f?

Bumpy?

Graph



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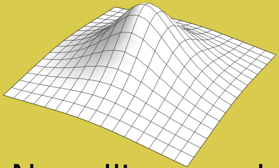
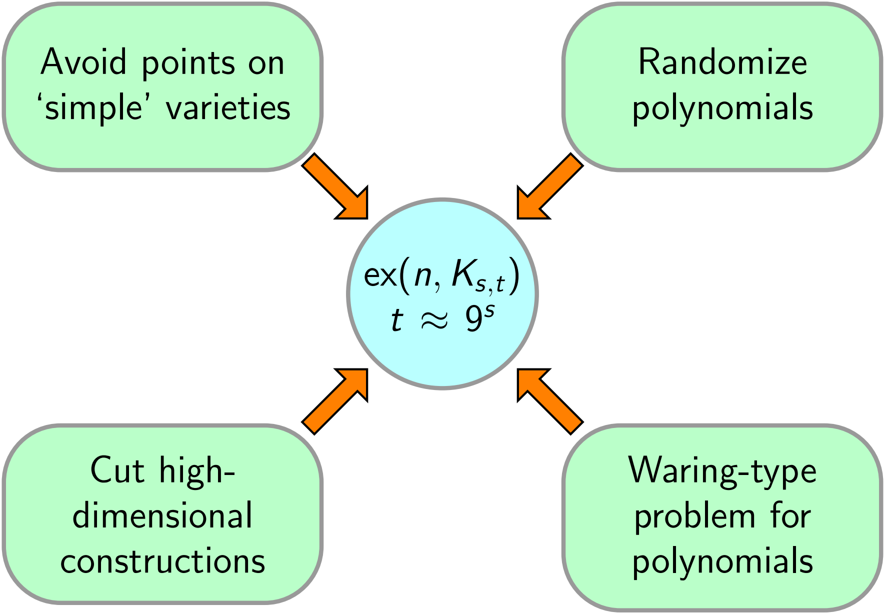


No collinear pts!

Full story


Avoid points on 'simple' varieties
Randomize polynomials
$\mathrm{ex}(n, K_{s,t})$
$t \approx 9^s$
Cut high-dimensional constructions
Waring-type problem for polynomials