Induced subgraphs of Ramsey graphs with many distinct degrees

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Abstract

An induced subgraph is called *homogeneous* if it is either a clique or an independent set. Let $\hom(G)$ denote the size of the largest homogeneous subgraph of a graph G. In this short paper we study properties of graphs on n vertices with $\hom(G) \le C \log n$ for some constant C. We show that every such graph contains an induced subgraph of order αn in which $\beta\sqrt{n}$ vertices have different degrees, where α and β depend only on C. This proves a conjecture of Erdős, Faudree and Sós.

1 Introduction

All graphs considered here are finite, undirected and simple. Given a graph G with vertex set $V(G)$ and edge set $E(G)$, let $\alpha(G)$ and $\omega(G)$ denote the size of the largest independent set and the size of the largest clique in G, respectively. An induced subgraph of G is called *homogeneous* if it is either a clique or an independent set. The size of the largest homogeneous subgraph of a graph G is denoted by $\hom(G)$, i.e., let $\hom(G) = \max\big(\alpha(G), \omega(G)\big)$. A classic result in Ramsey Theory [13] asserts that $\hom(G) \ge \frac{1}{2}\log n$ for any graph G on n vertices. On the other hand, it was shown by Erdős [7] that there are graphs with $\hom(G) \le 2\log n$. (Here, and throughout the paper all logarithms are to the base 2). The graphs for which $\hom(G)$ is very small compared to the number of vertices are usually called Ramsey graphs. The only kind of proof of existence of graphs with $\hom(G) \le O(\log n)$ which we have so far comes from various models of random graphs with edge density bounded away from 0 and 1. This supports the belief that any graph with small $\hom(G)$ looks 'random' in one sense or another. By now there are several known results which indeed show that Ramsey graphs have certain random-like properties.

The first advance in making the above intuition rigorous was made by Erdős and Szemerédi [14] who proved that $\hom(G) \le C\log n$ for fixed C implies that the edge density of G is indeed bounded away from 0 and 1. Later Erdős and Hajnal [10] proved that such graphs are *k-universal* for every fixed k, i.e., they contain every graph H on k vertices as induced subgraph. This was extended further by Prömel and Rödl [16], who obtained asymptotically best possible result. They proved that if $\hom(G) \le C\log n$ then G is an fact $c\log n$-universal for some constant c which depends on C.

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A related result which shows that Ramsey graphs have many distinct induced subgraphs was obtained by Shelah [17] (improving earlier estimate from [3]). He settled a conjecture of Erdős and Rényi, that every graph G with $\hom(G) \le C \log n$ contains 2^{cn} non-isomorphic induced subgraphs, where c is a positive constant depending only on C. Another question, with a similar flavor, was posed by Erdős and McKay [8, 9]. It asks whether every graph on n vertices with no homogeneous subgraph of order $C \log n$ contains an induced subgraph with exactly t edges for every $1 \le t \le \Theta(n^2)$. This conjecture is still wide open. In [5] it was proved for random graphs. In the general case, Alon, Krivelevich and Sudakov [2] proved that such graph always contains induced subgraphs with every number of edges up to n^δ, where $\delta > 0$ depends only on C.

In this paper we study the number of distinct degrees in induced subgraphs of Ramsey graphs. The following problem was posed by Erdős, Faudree and Sós [8, 9]. They conjectured that every G on n vertices with $\hom(G) \le C \log n$ contains an induced subgraph on a constant fraction of vertices which has $\Omega(\sqrt{n})$ different degrees. Here we obtain the result that confirms this conjecture and gives an additional evidence for the random-like behavior of Ramsey graphs.

Theorem 1.1 *Let G be a graph on n vertices with $\hom(G) \le C \log n$, for some constant C. Then G contains an induced subgraph of order αn with $\beta\sqrt{n}$ vertices of different degrees, where α and β depend only on C.*

Throughout this paper, we will make no attempt to optimize our absolute constants, and will often omit floor and ceiling signs whenever they are not crucial, for the sake of clarity of presentation. We also may and will assume that the number of vertices of G is sufficiently large.

2 Proof of the main theorem

First we define the notation that we are going to use. Numerous absolute constants that appear throughout the proof are denoted by $c_1, c_2, \ldots$. In a graph, the density of a set of vertices A is defined as

$$d(A) = \frac{e(A)}{\binom{|A|}{2}}$$

where $e(A)$ is the number of edges which are contained inside A. Similarly, for any two sets A and B the density is

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

where $e(A, B)$ is the number of edges between A and B. We abbreviate $d(\{x\}, B)$ to $d(x, B)$. For any vertex x and a set B the neighbors of x in B are denoted by $N_B(x)$ and non-neighbors are denoted by $\overline{N}_B(x)$. We denote the degree by $\deg_B(x) = |N_B(x)|$. For two sets X and Y, the usual symmetric difference $(X \setminus Y) \cup (Y \setminus X)$ is denoted by $X \Delta Y$.

In the proof we will use the following theorem of Erdős-Szemerédi mentioned above.

Theorem 2.1 ([14]) *Let G be a graph on n vertices of density $d \le 1/2$. Then there exist an absolute constant c_1 such that*

$$\hom(G) \ge c_1 \frac{1}{d\log(1/d)} \log n.$$

Our proof of the conjecture of Erdős and Sós is based on the following result, which merits being stated separately. First, we need a definition, though. A graph H of order n is called *c-diverse* if for every vertex $x \in V(H)$ there are at most $n^{1/5}$ vertices $y \in V(H)$ with $|N_H(x)\Delta N_H(y)| < cn$.

Lemma 2.2 *Let G be a graph on n vertices with $\hom(G) \le C\log n$. Let $K = \lceil c_2 C\log(C+1)\rceil$, $m = 8^{-K}n$ and $c = 1/K$. If $m \ge 4$ and an absolute constant c_2 is sufficiently large, then G contains an induced c-diverse subgraph on at least m vertices.*

Proof. To prove this statement we will use some ideas from [16, 17]. Suppose the conclusion of the lemma fails. We will construct a sequence of disjoint vertex sets $S_1, S_2, \dots, S_K$ each of the size $m^{1/5}$ such that for every $1 \le j \le K$ either

$$d(S_j, S_i) < 8c \quad \text{for all } i > j \tag{1a}$$

or

$$d(S_j, S_i) > 1 - 8c \quad \text{for all } i > j. \tag{1b}$$

Simultaneously with the sequence of S_i's we construct a nested sequence of induced subgraphs $G = G_0 \supset G_1 \supset \cdots \supset G_K$, which satisfy $|G_i| \ge |G_{i-1}|/8$ for all $1 \le i \le K$. The sets S_i will be chosen so that $S_i \subset G_{i-1}$.

Suppose the sets S_j as well as the graphs G_j have been constructed for all $j < i$ and we wish to construct S_i and G_i. The inductive hypothesis implies that $|G_{i-1}| \ge 8^{1-i}|G| \ge m$. Since the conclusion of the lemma fails, G_{i-1} is not c-diverse. Therefore, there exist $x \in V(G_{i-1})$ and a set $S_i \subset G_{i-1}$ with $|S_i| \ge m^{1/5}$ such that $|N_{G_{i-1}}(x)\Delta N_{G_{i-1}}(y)| \le c|G_{i-1}|$ for every $y \in S_i$. By throwing away elements of S_i if needed, we can assume that $|S_i| = m^{1/5}$. Let $B = V(G_{i-1}) \setminus S_i$. Since clearly $m^{1/5} \le m/2$, we have that $|B| \ge |G_{i-1}|/2$.

Suppose $|\overline{N}_B(x)| \ge |B|/2$. Then for every $y \in S_i$ we have $d(y, \overline{N}_B(x)) \le c|G_{i-1}|/|\overline{N}_B(x)| \le 4c$. Let $F = \{z \in \overline{N}_B(x) \mid d(z, S_i) \le 8c\}$ and $D = \overline{N}_B(x) \setminus F$. Since

$$\begin{aligned} 8c|D| &\le \sum_{z\in D} d(z, S_i) = \frac{e(D, S_i)}{|S_i|} \le \frac{e(\overline{N}_B(x), S_i)}{|S_i|} \\ &= \frac{|\overline{N}_B(x)|}{|S_i|} \sum_{y\in S_i} d(y, \overline{N}_B(x)) \le 4c|\overline{N}_B(x)|, \end{aligned}$$

we have that $|D| \le |\overline{N}_B(x)|/2$ and so $|F| \ge |\overline{N}_B(x)|/2 \ge |B|/4$. Similarly, if $|N_B(x)| \ge |B|/2$, then we set $F = \{y \in N_B(x) \mid d(y, S_i) \ge 1 - 8c\}$ and $|F| \ge |B|/4$ holds again. Let G_i be the graph induced on F. By the definition of the set F either all subsets $U \subseteq G_i$ satisfy satisfy $d(S_i, U) \le 8c$ or they all satisfy $d(S_i, U) \ge 1 - 8c$. This guarantees that for $t > i$ either all $d(S_i, S_t)$ will satisfy

(1a) or they all will satisfy (1b). Since $|B| \geq |G_{i-1}|/2$, we have $|G_i| \geq |G_{i-1}|/8$. This completes the inductive construction of S_i and G_i.

By passing to the complement if needed, we may assume that for at least half of all the sets $\{S_i\}_{i=1}^K$ the alternative (1a) holds. Recall that $c = 1/K$. Let $S_{i_1}, \ldots, S_{i_r}$ be the sets for which (1a) holds, where $r \geq K/2$. Let $S = \bigcup_{k=1}^r S_{i_k}$. It is easy to see that, $d(S) \leq 8c + 1/r \leq 10/K$. Therefore Theorem 2.1 implies that $\hom(G) \geq \hom(S) \geq c_1 \frac{K}{10\log(K/10)} \log|S|$. Since $\log|S| \geq \frac{1}{5}\log m \geq \frac{1}{5}\log n - K$, by choosing a large enough value of c_2, we get $\hom(G) \geq \hom(S) \geq 6C\log|S| > C\log n$ contradicting the assumption of the lemma. □

Remark. The constant 1/5 appearing in the definition of diverse graph can be replaced by any fixed $0 < \epsilon < 1$. In the case of such a replacement, the (far from optimal) constants appearing in the Lemma 2.2 will have to be changed accordingly.

Lemma 2.3 *Let G be a c-diverse graph on n vertices. Then for each $n/4 \leq m \leq 3n/4$ there is an induced subgraph of G on m vertices containing vertices of $\frac{1}{576}\sqrt{cn}$ distinct degrees.*

Proof. Fix m as above and choose a set A uniformly at random among all m-element subsets of $V(G)$. Let H be the graph induced on A. It is enough to show that with positive probability H has the desired property.

For distinct vertices $x, y \in V(G)$ define the indicator random variable I_{xy} by

$$I_{xy} = \begin{cases} 1, & \text{if } x, y \in V(H) \text{ and } \deg_H(x) = \deg_H(y), \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} E[I_{xy}] &= \Pr[x, y \in V(H)] \Pr[\deg_H(x) = \deg_H(y) \mid x, y \in V(H)] \\ &\leq \Pr[\deg_A(x) = \deg_A(y)]. \end{aligned}$$

Furthermore,

$$\Pr\big[\deg_A(x) = \deg_A(y)\big] = \Pr\big[|(N_G(x) \setminus N_G(y)) \cap A| = |(N_G(y) \setminus N_G(x)) \cap A|\big]$$

Let $D = N_G(x) \Delta N_G(y)$ and $k = |D \cap A|$. Suppose that $|D| \geq cn$ and let W be the event that $k < |D|/8$ or $k > 7|D|/8$. Since the value of k has hypergeometric distribution, it is sharply concentrated around its mean $|D|/4 \leq \frac{m}{n}|D| \leq 3|D|/4$. Thus, by Azuma-Hoeffding inequality (see, e.g., Chapter 2 of [15]), we have $\Pr[W] \leq e^{-\Theta(|D|)} = e^{-\Theta(n)} = o(n^{-1/2})$. Then

$$\Pr\big[\deg_A(x) = \deg_A(y) \mid |D|/8 \leq |D \cap A| = k \leq 7|D|/8\big] = \frac{\binom{|N_G(x) \setminus N_G(y)|}{k/2}\binom{N_G(y) \setminus N_G(x)}{k/2}}{\binom{|D|}{k}},$$

which by Stirling's formula together with convexity does not exceed

$$\frac{\binom{|D|/2}{k/2}^2}{\binom{|D|}{k}} \leq (1 + o(1))\sqrt{\frac{2|D|}{k(|D| - k)}} \leq \frac{7}{\sqrt{|D|}} \leq \frac{7}{\sqrt{cn}}.$$

Summing up over all values of k we conclude that if $|N_G(x)\Delta N_G(y)| \geq cn$ then $E[I_{xy}] \leq \frac{8}{\sqrt{cn}}$.

Let $I = \sum_{x,y\in V(G)} I_{xy}$. By the definition of c-diverse graph we have that $E[I_{xy}] \leq \frac{8}{\sqrt{cn}}$ for all but at most $n^{1+1/5}$ pairs $x, y \in V(G)$. Therefore $E[I] \leq n^{1+1/5} + 8n^2/\sqrt{cn} \leq 9n^{3/2}/\sqrt{c}$. Hence there is a particular choice for A such that $I \leq 9n^{3/2}/\sqrt{c}$. Let r be the number of vertices of distinct degrees in H and let a_k be the number of vertices in H with degree k. Then $\sum_k a_k = m$ and by convexity

$$I = \sum_k \binom{a_k}{2} \geq r\binom{\frac{1}{r}\sum_k a_k}{2} \geq \frac{m^2}{4r} \geq \frac{n^2}{64r}.$$

This implies $r \geq \frac{1}{9\cdot 64}\sqrt{cn}$ and completes the proof. □

The proof of Theorem 1.1 follows immediately from the two previous lemmas. It is worth noting that the value of β that it gives is $(C+1)^{-c_3 C}$ for some absolute constant c_3.

We have been unable to decide whether in Theorem 1.1 the exponent 1/2 in $n^{1/2}$ can be further improved to $1/2+\epsilon$ for some constant $\epsilon > 0$. However, using random graphs, one can show that 1/2 cannot be replaced by anything greater than 2/3. As usual, $G(n, 1/2)$ is the probability space of all labeled graphs on n vertices, where every edge appears randomly and independently with probability 1/2. We say that the random graph has a property $\mathcal{P}$ *almost surely*, or a.s. for brevity, if the probability that $G(n, 1/2)$ satisfies $\mathcal{P}$ tends to 1 as n tends to infinity. It is well known (see [7]) that the largest homogeneous subgraph of $G(n, 1/2)$ has a.s. size $O(\log n)$. Next we prove that an induced subgraph of this graph cannot have too many distinct degrees.

Proposition 2.4 *The random graph $G(n, 1/2)$ almost surely contains no induced subgraph with $8n^{2/3}$ vertices of distinct degrees.*

Proof. Let A be a subset of $G(n, 1/2)$ of size a such that the subgraph G' induced by A has $8n^{2/3}$ vertices with different degrees. Then either G' has at least $2n^{2/3}$ vertices with degree $\geq a/2 + 2n^{2/3}$ or it has at least $2n^{2/3}$ vertices with degree $\leq a/2 - 2n^{2/3}$. Consider the first case, the other one can be treated similarly. Let $B \subset A$ be the set of $b = 2n^{2/3}$ vertices whose degree in G' is at least $a/2 + 2n^{2/3}$. Since $\sum_{v\in B} \deg_A(v) = 2e(B) + e(B, A \setminus B)$, it is easy to see that, either there are at least $b^2/4 + bn^{2/3}/2$ edges inside B or there are at least $b(a-b)/2 + bn^{2/3}$ edges between B and $A \setminus B$. By Chernoff's inequality (see, e.g., Appendix A [4]) the probability of the first event is at most

$$\binom{n}{b} e^{-\Theta(n^{4/3})} \leq e^{O(n^{2/3}\log n)} e^{-\Theta(n^{4/3})} = o(1).$$

Similarly the probability of the second event is bounded by

$$\sum_a \binom{n}{b}\binom{n}{a} e^{-bn^{4/3}/(2a)} \leq n \cdot n^b \cdot 2^n \cdot e^{-n} = 2^{(1+o(1))n} e^{-n} = o(1).$$

Thus with probability $1 - o(1)$ there is no induced subgraph G' as above. □

It is worth noting that the exponent 2/3 in the above proof is essentially best possible, since $G(n, 1/2)$ a.s. contains an induced subgraph G' of order m which has $\Omega(n^{2/3})$ vertices of degree

$\geq m/2 + \Omega(n^{2/3})$. Indeed, fix an arbitrary set B of size $4n^{2/3}$ in $G(n, 1/2)$, and let

$$A = \left\{x \in V(G) \setminus B \mid \deg_B(x) \geq |B|/2 + n^{1/3}\right\}.$$

Using approximation of the binomial distribution by standard normal distribution it is easy to check that for each $x \notin B$, $\Pr[x \in A] \geq 1/10$. Since for distinct vertices these events are clearly independent, we have that a.s. $|A| \geq n/11$. Let

$$B_1 = \{y \in B \mid \deg_A(y) \geq |A|/2 + n^{2/3}/100\}$$

and $B_2 = B \setminus B_1$. Then, using that $|A| \geq n/11$ and $|B_2| \leq 4n^{2/3}$, we conclude

$$\begin{aligned} e(A, B_1) &= e(A, B) - e(A, B_2) \geq |A||B|/2 + |A|n^{1/3} - |A||B_2|/2 - |B_2|n^{2/3}/100 \\ &\geq |A||B_1|/2 + |A|n^{1/3}/2. \end{aligned} \tag{2}$$

Suppose that $|B_1| \leq n^{2/3}/40$, then by Chernoff bound the probability that $G(n, 1/2)$ contains sets A and B_1 satisfying inequality (2) is at most

$$\binom{n}{|B_1|}\binom{n}{|A|} e^{-|A_1|n^{2/3}/(8|B_1|)} \leq \left(\frac{en}{|A_1|}\right)^{(1+o(1))|A_1|} e^{-5|A_1|} \leq e^{4|A_1|}e^{-5|A_1|} = o(1).$$

This implies that a.s. $|B_1| > n^{2/3}/40$. Since a.s. most $v \in B_1$ satisfy $\deg_{B_1}(v) \geq |B_1|/2 - n^{1/2}$ we have that the subgraph induced by $B_1 \cup A$ has the desired property.

3 Concluding remarks

- We already mentioned in introduction several results and conjectures about properties of Ramsey graphs. An additional such problem, which is closely related to our results, was posed by Erdős, Faudree and Sós [8, 9]. They conjectured that every graph on n vertices with no homogeneous subset of size $C \log n$ contains at least $\Omega\big(n^{5/2}\big)$ induced subgraphs any two of which differ either in the number of vertices or in the number of edges. Using our proof of Theorem 1.1 one can easily obtain the following result.

  Proposition 3.1 *If G has n vertices and $\hom(G) \leq C \log n$, then the number of distinct pairs $\big(|V(H)|, |E(H)|\big)$ as H ranges over all induced subgraphs of G is at least $\Omega\big(n^{3/2}\big)$.*

  Proof. By Lemma 2.2, G contains an induced c-diverse subgraph G' on $n' = \Omega(n)$ vertices. By Lemma 2.3, G' contains for every $n'/4 \leq m \leq 3n'/4$ an induced subgraph on m vertices with $\Omega(\sqrt{n'}) = \Omega(\sqrt{n})$ distinct degrees. Deleting these vertices, one at a time, gives $\Omega(\sqrt{n})$ induced subgraphs of G of order $m - 1$ all of which obviously have different numbers of edges. Since there are $\Omega(n)$ choices for m the result follows. □

  Although this proposition is much weaker than Erdős-Faudree-Sós conjecture, we believe that our methods might prove useful to attack their problem.

- The conventional belief that graphs with small homogeneous subgraphs have many random-like properties goes beyond the graphs with $\hom(G) \le C \log n$. For example, the famous Erdős-Hajnal conjecture [11] states that for every fixed graph H, there exist $\epsilon(H) > 0$ such that every graph on n vertices without homogeneous subgraphs of order n^ϵ contains an induced copy of H.

  Even graphs with relatively large homogeneous subgraphs tend to be jumbled. For instance, Alon and Bollobás [1] proved that for sufficiently small δ if a graph contains no homogeneous set on $(1-4\delta)n$ vertices, then the graph contains δn^2 non-isomorphic induced subgraphs. A somewhat stronger result was proved by Erdős and Hajnal [12].

  In the light of the above, it seems likely that every graph without homogeneous subgraph of order n^ϵ should contain an induced subgraph of linear size with $\Omega(n^{1/2-\epsilon})$ distinct degrees. However, our proof of Theorem 1.1 does not seem to extend to show this.

- A tournament with no directed cycles is called *transitive*. Let $\mathrm{tran}(T)$ be the number of vertices in the largest transitive subtournament of T. It is well known [6, 18] that every n-vertex tournament contains a transitive subtournament of order at least $c \log n$, and this result is tight apart from the value of the constant. Similarly, we call a tournament Ramsey if $\mathrm{tran}(T)$ is very small compared to the number of vertices of T. Our technique can be used to prove the following analogue of Theorem 1.1.

  Theorem 3.2 *Let T be a tournament on n vertices with $\mathrm{tran}(T) \le C \log n$, for some constant C. Then T contains a subtournament of order αn and $\beta\sqrt{n}$ vertices of different outdegrees, where α and β depend only on C.*

  Sketch of proof. To prove this result we need a version of Erdős-Szemerédi theorem for tournaments. The version we use states that if there is an ordering of the vertices of the tournament on n vertices with only $d\binom{n}{2}$ edges going in the reverse direction to the ordering, then
  $$\mathrm{tran}(T) \ge c_1 \frac{1}{d \log(1/d)} \log n.$$
  To show this, consider a graph G on the same vertex set as T whose edges are the edges of T that go in the reverse direction to the ordering. Then the density of the graph G is d, and therefore Erdős-Szemerédi theorem guarantees existence of a homogeneous subgraph of G of size $c_1 \frac{1}{d\log(1/d)} \log n$. Since every homogeneous subgraph in G is a transitive subtournament of T, this establishes Erdős-Szemerédi theorem for tournaments.

  Define diverse tournaments in the same way as diverse graphs with neighborhoods replaced by outneighborhoods. With this definition, the rest of the proof (Lemmas 2.2 and 2.3) carries over to the case of tournaments. We omit the details. □

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