A point in many triangles

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Abstract

We give a simpler proof of the result of Boros and Füredi that for any finite set of points in the plane in general position there is a point lying in 2/9 of all the triangles determined by these points.

Introduction

Every set P of n points in $\mathbb{R}^d$ in general position determines $\binom{n}{d+1}$ d-simplices. Let p be another point in $\mathbb{R}^d$. Let $C(P,p)$ be the number of the simplices containing p. Boros and Füredi [2] constructed a set P of n points in $\mathbb{R}^2$ for which $C(P,p) \leq \frac{2}{9}\binom{n}{3} + O(n^2)$ for every point p. They also proved that there is always a point p for which $C(P,p) \geq \frac{2}{9}\binom{n}{3} + O(n^2)$. Here we present a new simpler proof of the existence of such a point p.

Proof

Let P be a set of n points in the plane. By the extension of a theorem of Buck and Buck [3] due to Ceder [4] there are three concurrent lines that divide the plane into 6 parts each containing at least $n/6 - 1$ points in its interior. Denote by p the point of intersection of the three lines. Every choice of six points, one from each of the six parts, determines a hexagon containing the point p.

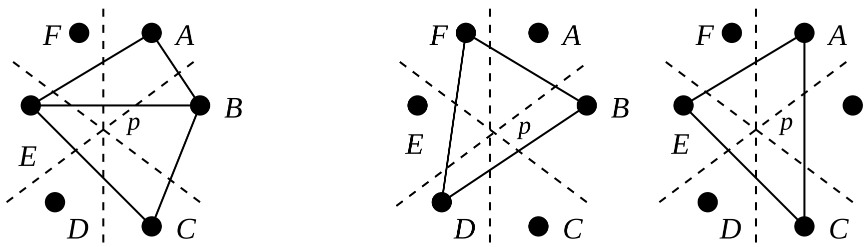

Figure 1: a) $p \in ABE$ or $p \in BCE$ b) $p \in ACE$ and $p \in BDF$

Among the $\binom{6}{3} = 20$ triangles determined by the vertices of the hexagon, at least 8 triangles contain the point p. Indeed, from each of the six pairs of triangles situated as in Figure 1a we get one triangle containing p. In addition, p is contained in both triangles of the Figure 1b. Therefore, by double counting, the number of triangles containing p is at least

$$\frac{8(n/6-1)^6}{(n/6-1)^3} = \frac{2}{9}\binom{n}{3} + O(n^2).$$

For the sake of completeness we include a sketch of a proof of the modification of the theorem of Buck and Buck that we used above.

Proposition 1. *Let μ be a finite measure absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$. Then there are three concurrent lines that partition the plane into six parts of equal measure.*

The partition theorem for the finite set of point P follows by letting μ be the restriction of the Lebesgue measure to the union of tiny disks of equal size centered at the points of P. Since P is in general position, none of the three lines passes through more than two of the disks.

Proof sketch. The given measure can be made into one which gives every open set a strictly positive measure, and which differs little from the given one. Proving the result for the latter, and using a compactness argument, one is through. Hence we can assume the property mentioned, and we normalize the total measure of the plane to 1.

Let now u be a unit vector. There is a unique directed line $L(u)$ pointing in the direction u and cutting the plane in two parts of measure 1/2. For any point P on $L(u)$ there are six unique rays from P, denoted $A(u,P),\dots,F(u,P)$ in clockwise order, splitting the plane in sectors of measure 1/6, with $A(u,P)$ in the direction u. Note that $L(u)$ is the union of $A(u,P)$ and $D(u,P)$. When P moves along $L(u)$ in the direction u, the ray $B(u,P)$ will turn counterclockwise in a continuous way, becoming orthogonal to $L(u)$ at some point. As the clockwise turning $E(u,P)$ behaves in the same way, there will be a unique $P^*(u)$ such that $B(u,P^*(u))$ and $E(u,P^*(u))$ form a line.

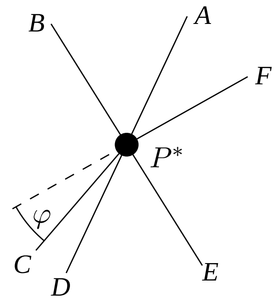

Figure 2: Six rays

The line L, the point P^* and the six rays from P^* clearly depend continuously on u. In particular the angle $\varphi(u)$ one must turn $C(u,P^*(u))$ counterclockwise to complete $F(u,P^*)$ to a line varies continuously. But for any u, we have $C(-u,P^*(-u)) = F(u,P^*(u))$, and hence $\varphi(-u) = -\varphi(u)$. This shows that for some v the angle $\varphi(v)$ vanishes and the rays $C(v,P^*(v))$ and $F(v,P^*(v))$ form a line. This finishes the proof. $\square$

For no dimension higher than 2 the optimal bounds for $C(P,p)$ are known. Bárány [1] showed that there is always a point p for which $c(P,p) \geq (d+1)^{-d}\binom{n}{d+1} + O(n^d)$.

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References

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