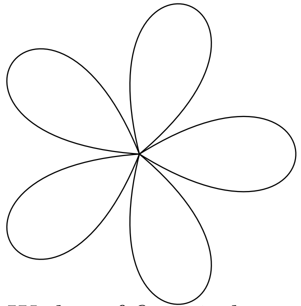
Topological methods in combinatorics: on wedges and joins*

A *pointed topological space* is a pair (X, x_0) consisting of a topological space X and a point $x_0 \in X$. *Wedge sum* of two pointed spaces (X, x_0) and (Y, y_0) is a pointed topological space (Z, z_0) where Z is the quotient space $Z = X \sqcup Y / \{x_0 = y_0\}$, where $\sqcup$ denotes the disjoint union, and z_0 is the point obtained by identification of x_0 with y_0. The wedge sum is written as $(X, x_0) \vee (Y, Y_0)$, or simply as $X \vee Y$. The latter notation is generally misleading, as we do not explicitly specify the basepoints x_0 and y_0 in X and Y. Choosing different basepoints x_0, y_0 in X and Y generally gives non-homeomorphic spaces for $X \vee Y$. However, there are two reasons why changing basepoints usually often does not matter. First, if (X, x_0) and (X, x_0') are two pointed spaces that differ only in the choice of the basepoint, and there is a homeomorphism $h \colon X \to X$ such that $h(x_0) = x_0'$, then changing the basepoint does not change the homeomorphism type of $X \vee Y$.



Wedge of five circles.

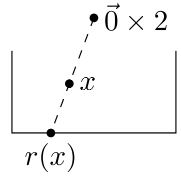
Second, if X is a simplicial complex and the points x_0 and x_0' are connected by a path, then the homotopy type of $X \vee Y$ does not change with the change of the basepoint. Whereas it easy to convince oneself by imagining moving the contact point between X and Y along the path connecting x_0 and x_0', the actual proof requires more, since the result is false if X is an arbitrary topological space. To prove this, we need the property of geometric simplicial complexes, which says that we can extend a homotopy of subcomplex to the full complex.

Definition. *Suppose X is a topological space, and A is a subspace of X. Then the pair (X, A) has* homotopy extension property *if for every space Y and every map $f \colon X \to Y$ and a homotopy $\bar{F} \colon A \times [0,1] \to Y$ satisfying $\bar{F}(\cdot, 0) = f|_A$ there is an extension of $\bar{F}$ to a homotopy $F \colon X \times [0,1] \to Y$ satisfying $F(\cdot, 0) = f$.*

Lemma 1. *If L is a simplicial complex, and L' is a subcomplex of L. Then the pair $(|L|, |L'|)$ has the homotopy extension property.*

Proof. It suffices to deal with the case when $L \setminus L'$ consists of a single simplex. Indeed, we can then extend to simplices in $L \setminus L'$ one at a time starting from the simplices of the smallest dimension and working upward.

So, let Δ be unique simplex in $L \setminus |L'|$. Let $K = (\partial\Delta \times [0,1]) \cup (\Delta \times \{0\})$. By the assumption, $\bar{F}$ is defined on $|K|$ Think of $|\Delta|$ as a geometric simplex living in $\mathbb{R}^n$ with the centre at the origin. Then $|\Delta| \times [0,1]$ lives in $\mathbb{R}^n \times \mathbb{R}$. For each point $x \in |\Delta| \times [0,1]$ let $r(x)$ be the intersection point of $|K|$ with the ray originating at $\vec{0} \times 2 \in \mathbb{R}^n \times \mathbb{R}$ and going through x. Then we can define the desired extension by $F(x) = \bar{F}(r(x))$. □



Note that the proof of the Lemma 1 gives more, namely that the homotopy of $|L \times [0,1]|$ and $|L' \times [0,1] \cup L \times \{0\}|$. Indeed, composition of maps r constructed for each simplex of $L \setminus L'$ gives a retraction map $\phi \colon |L \times [0,1]| \to |L' \times [0,1] \cup L \times \{0\}|$.

Proposition 2. *Suppose (X, x_0) is a pointed simplical complex, Y a topological space, and $y_0, y_1 \in Y$ are two points in the same path-connected component of Y. Let Z_0 be the underlying topological space of $(X, x_0) \vee (Y, y_0)$, and let Z_1 be the underlying topological space of $(X, x_0) \vee (Y, y_1)$. Then $Z \simeq Z'$.*

Proof. Since y_0, y_0' are in the same path-connected component, there is a function $f \colon \{x_0\} \times [0,1] \to Y$ such that $f(x_0, 0) = y_0$ and $f(x_0, 1) = y_1$. Define the equivalence relation $\approx$ on $(X \times [0,1]) \sqcup Y$ by

$$(x, t) \approx y \text{ if and only if } x = x_0, \text{ and } f(x_0, t) = y.$$

Introduce the spaces

$$\begin{aligned} W &= (X \times [0,1]) \sqcup Y / \approx, \\ W_0 &= (\{x_0\} \times [0,1] \cup X \times \{0\}) \sqcup Y / \approx, \\ W_1 &= (\{x_0\} \times [0,1] \cup X \times \{1\}) \sqcup Y / \approx, \end{aligned}$$

Let $\phi \colon X \times [0,1] \to \{x_0\} \times [0,1] \cup X \times \{0\}$ be the retraction constructed in Lemma 1. Then ϕ induces the map $\Phi \colon W \to W_0$ given by

$$\begin{aligned} (x,t) &\mapsto \phi(x,t) & \quad (x,t) &\in X \times [0,1] \\ y &\mapsto y & y &\in Y. \end{aligned}$$

The map Φ is a retraction of W onto W_0. So, W_0 and W are homotopic. Since W_0 is homeomorphic to Z_0, it follows that W and Z_0 are homotopic. Similarly for W and Z_1. As, Z_0 and Z_1 are homotopic to the same space, the proof is complete. □

Another useful consequence of homotopy extension property is that the homotopy type is invariant under quotients by contractible subcomplexes:

Proposition 3. *Suppose L is a simplicial complex, and L' is a subcomplex. Suppose further that L' is contractible (i.e., homotopy equivalent to a one-point space). Then $|L|/|L'|$ is homotopy equivalent to L.*

*These notes are from `http://www.borisbukh.org/TopCombLent12/notes_wedgejoin.pdf`.

Proof. Let $\bar{F}\colon |L'|\times[0,1]\to|L'|$ be the contraction, i.e., $\bar{F}(x,0)=x$ for all $x\in|L'|$ and $\bar{F}(\cdot,1)=p$ for some $p\in|L'|$. The map $\bar{F}$ naturally to $|L'|\times[0,1]\cup L\times\{0\}\to|L|$ by putting $\bar{F}(x,0)=x$ for all $x\in|L|$. By Lemma 1 there is an extension of $\bar{F}$ to $F\colon |L|\times[0,1]\to|L'|$. Let $f\colon |L|/|L'|\to|L|$ be given by

$$f(x)=\begin{cases}F(x,1) & \text{if } x\neq[L'],\\ p & \text{if } x=[L'].\end{cases}$$

Then f is a continuous. Since $\pi\circ f=\mathrm{id}_{|L|/|L'|}$ and $f\circ\pi\simeq\mathrm{id}_{|L|}$ via the homotopy F, it follows that $|L|/|L'|$ and $|L|$ are homotopic. ☐

A *reduced join* of pointed space (X,x_0) and (Y,y_0) is the space

$$X*'Y=(X*Y)/\big(\{x_0\}*Y\cup X*\{y_0\}\big).$$

Lemma 4. *If (X,x_0) and (Y,y_0) are pointed simplicial complexes, then $|X|*'|Y|\simeq|X*Y|$.*

Proof. The space $A=\{x_0\}*Y$ is contractible by pushing all the points towards x_0. Similarly for $B=X\times\{y_0\}$. Both $A=\{x_0\}*Y$ and $B=X*\{y_0\}$ are contractible subcomplexes of $X*Y$. Since $A\cap B=\{x_0\}*\{y_0\}$ is contractible, it follows that $A\cup B$ is contractible (exercise!). Hence, the lemma follows from Proposition 3. ☐

We can make reduced join $X*'Y$ into a pointed topological space by declaring $[\{x_0\}*Y\cup X*\{y_0\}]$ to be the basepoint. With this in mind, the following lemma is just an exercise in definition-chasing.

Lemma 5. *Suppose (X,x_0), (Y,Y_0) and (Z,z_0) are pointed topological spaces. Then the spaces $(X\vee Y)*'Z$ and $(X*'Z)\vee(Y*'Z)$ are homeomorphic.*

Proof. Both spaces are equal to

$$\{\lambda w\oplus(1-\lambda)y : w\in X\sqcup Y\}/\big(\{x_0,y_0\}*Z\cup(X\sqcup Y)*\{z_0\}\big).$$

☐

Corollary 6. *If (X,x_0), (Y,y_0) and (Z,z_0) are pointed simplicial complexes, then $(X\vee Y)*Z\simeq(X*Z)\vee(Y*Z)$.*

The Corollary 6 reduces the computation of the join of discrete spaces to a mindless manipulation:

Theorem 7. *Let $n\geq1$ and $m\geq2$ be integers. Suppose $K=[m]$ is an m-point discrete space. Then $K^{*(n+1)}$ is a wedge of $(m-1)^{n+1}$ n-spheres.*

Proof. We can write $[m]$ as a wedge of $m-1$ two-point spaces $[m]=[2]\vee\cdots\vee[2]$. Since $[2]^{*(n+1)}\cong S^n$, the result follows from Corollary 6. ☐