Matrix Theory: homework #9*
Due 30 October 2013, at start of class

1. Prove that two subspaces V_1 and V_2 of a vector space V are linearly independent if and only if $V_1 \cap V_2 = \{\vec{0}\}$.

2. Prove that
$$|(\vec{v}, \vec{u})| = \|\vec{v}\| \cdot \|\vec{u}\|$$
if and only if one of $\vec{v}$ and $\vec{u}$ is a multiple of another. (Hint: Analyze the proof of Cauchy–Schwarz inequality).

3. (a) Suppose $T\colon V \to V$ is a diagonalizable linear transformation on a complex finite-dimensional vector space V. Prove that there is a linear transformation $S\colon V \to V$ such that $S^2 = T$.
   (b) Give an example of a linear transformation T on a complex vector space such that there is no transformation S satisfying $S^2 = T$.

4. True or false: if A is a non-zero matrix, then there is a non-constant polynomial f such that $f(A) \neq 0$, and $f(A)$ is diagonalizable. Justify.

5. Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for an inner product space V. Suppose $\vec{u}, \vec{w} \in V$ and $(\vec{v}_i, \vec{u}) = (\vec{v}_i, \vec{w})$ for all $i = 1, \ldots, n$. Prove that $\vec{u} = \vec{w}$.

6. (Bonus problem) Suppose $T\colon V \to V$ is a linear transformation on a finite-dimensional complex vector space V such that for each $\vec{v} \in V$ the vectors $\vec{v}, T\vec{v}, T^2\vec{v}, \ldots, T^n\vec{v}$ are linearly dependent. Prove that linear transformations $I, T, T^2, \ldots, T^n$ are linearly dependent (partial credit will be awarded for the $n = 1$ case).

*This homework is from `http://www.borisbukh.org/MatrixTheory13/hw9.pdf`.