Matrix Theory: homework #8*
Due 23 October 2013, at start of class

1. Let $\mathcal{P}$ be the vector space that consists of all polynomials with real coefficients (with normal operations of additions and scalar multiplication). Let $T\colon \mathcal{P} \to \mathcal{P}$ be the linear transformation given by $f(x) \mapsto xf'(x)$. Find all the eigenvectors of T.

2. (a) Let $n \geq 1$ be an integer. Suppose $A, B \in M_{n,n}(\mathbb{C})$ are two complex matrices, and B is invertible. Prove that there is a scalar α such that $A + \alpha B$ is not invertible.

   (b) Find two real matrices A, B such that $A + \alpha B$ is invertible for every $\alpha \in \mathbb{R}$.

   (c) Let $n \geq 1$ be an *odd* integer. Suppose $A, B \in M_{n,n}(\mathbb{R})$ are two real matrices, and B is invertible. Prove that there is a scalar α such that $A + \alpha B$ is not invertible.

3. By diagonalizing an appropriate matrix, find a closed form expression for the n'th term of a sequence $a_0, a_1, a_2, \ldots$ that is given by $a_0 = 1$, $a_1 = 1$ and $a_n = 2a_{n-1} + 2a_{n-2}$.

4. Let $T\colon V \to V$ be an invertible linear transformation on a finite-dimensional vector space V. Prove that eigenspace of T corresponding to the eigenvalue λ is same as the eigenspace of T^{-1} corresponding to the eigenvalue λ^{-1}.

5. Let
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$
Give a formula for A^n in terms of n.

6. (Bonus problem) Let $A \in M_{n,n}(\mathbb{C})$ be a matrix with characteristic polynomial $(\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$. Note that some of λ_i's might be equal. Prove that $\operatorname{trace}(A^2) = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2$.

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*This homework is from `http://www.borisbukh.org/MatrixTheory13/hw8.pdf`.