Matrix Theory: homework #7*
Due 14 October 2013, at start of class

1. Compute, for each positive integer n, the determinants of the following n-by-n matrices

$$\text{a)} \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix} \qquad \text{b)} \begin{pmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & & \vdots \\ n(n-1)+1 & n(n-1)+2 & \cdots & n^2 \end{pmatrix}$$

2. For n-by-n matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{pmatrix}$$

   compute $\det(A+tI_n)$. You should get a nice expression involving $a_0, a_1, \ldots, a_{n-1}, t$. Row expansion and induction is probably the best way to go.

3. Do the exercise 5.6 on page 95 of the textbook.

4. (a) The formula for the determinant of 3-by-3 matrix as a sum over permutations has $3! = 6$ terms. Show that it is not possible for all six terms to be positive.

   (b) Let $n \geq 3$ be any integer. Prove that in the formula for the determinant of an n-by-n matrix as a sum of $n!$ terms, it is not possible for each term to be positive. (Hint: use part (a).)

*This homework is from `http://www.borisbukh.org/MatrixTheory13/hw7.pdf`.

5. (Bonus problem) Let $F(a_1, \ldots, a_n)$ be the determinant of

$$\begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & a_2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & a_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & a_n \end{pmatrix}.$$

Prove that

$$\frac{F(a_1, a_2, \ldots, a_n)}{F(a_2, a_3, \ldots, a_n)} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_n}}}$$