Matrix Theory: homework #6*
Due 9 October 2013, at start of class

1. A square matrix is called *nilpotent* if $A^k = 0$ for some positive integer k. Prove that $\det A = 0$ for every nilpotent matrix A.

2. Suppose A and B are similar n-by-n matrices.
   (a) Prove that $\operatorname{trace} A = \operatorname{trace} B$.
   (b) Prove that $\det(A + \alpha I_n) = \det(B + \alpha I_n)$ for every scalar α.
   (c) Give an example of matrices C and D such that $\operatorname{trace} C = \operatorname{trace} D$, and $\det C = \det D$, but C and D are not similar.

3. Suppose n-by-n invertible matrices A, B satisfy $AB = -BA$. Show that n must be even.

4. (a) Let A be a n-by-n matrix. Show that the $(n+m)$-by-$(n+m)$ block matrices
   $$\begin{pmatrix} I_m & * \\ 0 & A \end{pmatrix}, \quad \begin{pmatrix} A & * \\ 0 & I_m \end{pmatrix}, \quad \begin{pmatrix} I_m & 0 \\ * & A \end{pmatrix}, \quad \begin{pmatrix} A & 0 \\ * & I_m \end{pmatrix}$$
   all have determinant equal to $\det A$. Here $*$ can be anything.
   (b) Show that if A, B and C are n-by-n matrices, then $\det \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right) = \det A \det C$. (Hint: $\left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right) = \left(\begin{smallmatrix} I & B \\ 0 & C \end{smallmatrix}\right)\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$)
   (c) Suppose A is an n-by-m matrix, whereas B is an m-by-n matrix. Show that $\det(I_n - AB) = \det(I_m - BA)$. (Hint: evaluate $\det \left(\begin{smallmatrix} I_n & A \\ B & I_m \end{smallmatrix}\right)$ in two different ways.)

5. If σ is an odd permutation, explain why σ^2 is even, but σ^{-1} is odd.

6. (Bonus problem) Suppose $T_1, \dots, T_m$ are linear transformations from an n-dimensional space V to itself. Suppose $m > n$ and suppose that $T_1 + \cdots + T_m$ is invertible. Show one can remove at least one of the $T_1, \dots, T_m$ from the sum so that the sum of the remaining linear transformations is still invertible. (Solving the case when $m = 3$ and $n = 2$ gets partial credit.)

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*This homework is from `http://www.borisbukh.org/MatrixTheory13/hw6.pdf`.