Matrix Theory: homework #5*
Due 2 October 2013, at start of class

1. Let $T\colon V \to V$ be a linear transformation from a vector space V into itself. Let $\vec{v} \in V$ be a vector such that $T^m(\vec{v}) = \vec{0}$ for some positive integer m, but such that $T^{m-1}(\vec{v})$ is non-zero. What is the dimension of $\operatorname{span}\{\vec{v}, T(\vec{v}), T^2(\vec{v}), \ldots, T^{m-1}(\vec{v})\}$?

2. Suppose A is a matrix, and $A\vec{x} = \vec{0}$ has unique solution. Prove that $A^T\vec{x} = \vec{b}$ has a solution for every possible right side $\vec{b}$.

3. Suppose $S\colon V \to V$ and $T\colon V \to V$ are linear transformation from a finite-dimensional vector space V into itself.
   (a) Is $\operatorname{rank}(ST) = \operatorname{rank}(TS)$ always true? Justify.
   (b) Prove that $\operatorname{rank}(ST) \geq \operatorname{rank}(S) + \operatorname{rank}(T) - \dim V$.

4. Suppose $T\colon V \to V$ is a linear transformation from a finite-dimensional linear space into itself. Show that the following two assertions are equivalent:
   (a) $V = \operatorname{Ran}(T) + \operatorname{Ker}(T)$.
   (b) $\operatorname{Ran}(T) \cap \operatorname{Ker}(T) = \{\vec{0}\}$.

5. Suppose $T\colon V \to V$ is a linear transformation from a finite-dimensional linear space into itself.
   (a) Show that $\operatorname{rank}(T) = \operatorname{rank}(T^2)$ implies that $\operatorname{Ran}(T) \cap \operatorname{Ker}(T) = \{\vec{0}\}$.
   (b) Prove that there is a k such that $\operatorname{Ran}(T^k) \cap \operatorname{Ker}(T^k) = \{\vec{0}\}$.

6. (Bonus problem) Suppose $T_1, T_2, T_3\colon V \to V$ are three linear transformations such that $\operatorname{rank}(T_1 + T_2 + T_3) = 2$. Does it necessarily follow that one of $\operatorname{rank}(T_1 + T_2), \operatorname{rank}(T_1 + T_3), \operatorname{rank}(T_2 + T_3)$ must be at least 2?

*This homework is from `http://www.borisbukh.org/MatrixTheory13/hw5.pdf`.