Matrix Theory: homework #4*
Due 25 September 2013, at start of class

1. Suppose V is a subspace of $\mathbb{R}^n$. Show that there is a subspace U of $\mathbb{R}^n$ such that $V \cap U = \{\vec{0}\}$ and $U + V = \mathbb{R}^n$. (For notation $U + V$, see problem 5 on homework #3.)

2. Let $T: V \to W$ be a linear transformation between finite-dimensional spaces, and suppose $T\vec{x} = \vec{0}$ has a unique solution. Show that T is left-invertible.

3. (a) Suppose $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is a set of linearly independent vectors in a vector space V, and suppose that the set $\{\vec{w}_1, \ldots, \vec{w}_n\}$ is a basis for V. Show that if $n > m$, then there is an i such that the set $\vec{v}_1, \ldots, \vec{v}_m, \vec{w}_i$ is linearly independent.

   (b) Use (a) to deduce that if $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is linearly independent, and the set $\{\vec{w}_1, \ldots, \vec{w}_n\}$ is a basis, then one can find $n-m$ vectors among $\vec{w}_1, \ldots, \vec{w}_n$ such that these vectors together with $\vec{v}_1, \ldots, \vec{v}_m$ form a basis.

4. If $f(t) = a_N t^N + \cdots + a_1 t + a_0$ is a polynomial of degree N, and $B \in M_{n,n}$ is a square matrix, then $f(B)$ is defined to be $f(B) = a_N B^N + \cdots + a_1 B + a_0 I$, where B^p is the p'th power of B and I is the identity matrix.

   Prove that for every matrix $B \in M_{n,n}$ there is a non-zero polynomial f of degree at most n^2 such that $f(B) = 0$ (the zero matrix).

5. (Bonus problem) Suppose $V_1, \ldots, V_n$ are 2-dimensional subspaces of a vector space V. Suppose further that every two of these subspaces intersect non-trivially, i.e., $V_i \cap V_j \neq \{\vec{0}\}$ for all i, j. Prove that at least one of the following two statements must be true about $V_1, \ldots, V_n$:

   (a) There is a subspace U of V of dimension at most 3 such that U contains each of $V_1, \ldots, V_n$.

   (b) There is a subspace W of V of dimension 1 such that each of $V_1, \ldots, V_n$ contains W.

*This homework is from `http://www.borisbukh.org/MatrixTheory13/hw4.pdf`.