Matrix Theory: homework #3*
Due 16 September 2013, at start of class

1. Suppose $S\colon U \to W$ and $T\colon V \to U$ are two linear transformations, and ST is invertible. Show that S is right-invertible, and that T is left-invertible. (Hint: it is possible to write down the inverses explicitly).

2. Suppose $T\colon M_{n,n}(\mathbb{R}) \to \mathbb{R}$ is a linear transformation such that for every two n-by-n matrices A and B we have $T(AB) = T(BA)$. Prove that there is a scalar $\alpha \in \mathbb{R}$ such that $T(C) = \alpha \operatorname{trace} C$ for every $C \in M_{n,n}(\mathbb{R})$.

3. Let A be a symmetric matrix (i.e. it satisfies $A^T = A$). Suppose further that A is an invertible matrix. Is the inverse of A symmetric? Justify.

4. (a) Suppose U_1 and U_2 are subspaces of a vector space U. Show that $U_1 \cap U_2$ is also a subspace of U.

   (b) Suppose $U_1, U_2, U_3, \dots$ are each a subspace of a vector space U. Show that the intersection $U_1 \cap U_2 \cap U_3 \cap \cdots$ is also a subspace of U. (The intersection consists of all vectors that belong to all of $U_1, U_2, U_3, \dots$.)

5. (a) Suppose V and U are two subspaces of W. Show that the set $V + U \stackrel{\text{def}}{=} \{\vec{v} + \vec{u} : \vec{v} \in V,\ \vec{u} \in U\}$ is a subspace of W.

   (b) Show that if $\vec{v}_1, \dots, \vec{v}_m$ is a basis for V and $\vec{u}_1, \dots, \vec{u}_n$ is a basis for U, then there is a basis for $U + V$ that consists of at most $m + n$ vectors.

6. (Bonus problem) Suppose $\vec{v}_1, \dots, \vec{v}_n$ is a basis for the vector space V over field of scalars F. Let U be the vector space consisting of all linear functions from V to F. Find a basis for U.

*This homework is from `http://www.borisbukh.org/MatrixTheory13/hw3.pdf`.