Matrix Theory: homework #2*
Due 11 September 2013, at start of class

1. Prove or give a counterexample for each of following assertions:

   (a) If vectors \( \vec{v}_1, \ldots, \vec{v}_n \) are linearly independent, whereas vectors \( \vec{v}_1, \ldots, \vec{v}_n, \vec{w} \) are linearly dependent, then \( \vec{w} \) is a linear combination of \( \vec{v}_1, \ldots, \vec{v}_n \).

   (b) If \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) are three vectors such that both \( \{\vec{v}_1, \vec{v}_2\} \) and \( \{\vec{v}_1, \vec{v}_3\} \) are linearly dependent, then \( \{\vec{v}_2, \vec{v}_3\} \) is linearly dependent as well.

2. Let \( V \) consist of all infinite sequences \( a_0, a_1, a_2, a_3, a_4 \ldots \) of numbers such that \( a_{n+2} = 5a_{n+1} - 6a_n \) for all integers \( n \geq 0 \).

   (a) Show that \( V \) is a vector space;

   (b) Find two sequences, each of which is of the form \( a_n = c^n \), that are both in \( V \);

   (c) Show that the two sequences you found above form a basis for \( V \);

   (d) Find a formula for the \( n \)'th term of the sequence that starts 1, 4, 14, 46, \ldots.

3. A **magic square** (of order 3) is a 3-by-3 array of numbers such that the sum of every row, every column, and every big diagonal is the same. For example,

\[
\begin{bmatrix}
6 & 1 & 8 \\
7 & 5 & 3 \\
2 & 9 & 4
\end{bmatrix}
\]

is a magic square. Let \( V \) consist of all magic squares endowed with operations of component-wise addition, and component-wise scalar multiplication.

Show that \( V \) is a vector space, and find a basis for \( V \).

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*This homework is from [http://www.borisbukh.org/MatrixTheory13/hw2.pdf](http://www.borisbukh.org/MatrixTheory13/hw2.pdf)
4. An square matrix $A$ is called *upper triangular* if all entries lying below the diagonal are zero (i.e., $A_{ij} = 0$ whenever $i > j$). Suppose further that the diagonal entries are non-zero (i.e., $A_{ii} \neq 0$ for all $i$). Show that the columns of $A$ are linearly independent.

5. Do there exist three matrices $A, B, C$ such that $AB = BA$ and $AC = CA$, but $BC \neq CB$?

6. (Bonus problem) Vectors $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent. Vectors $\vec{w}_1, \ldots, \vec{w}_n$ form a basis, and so are vectors $\vec{w}_1, \ldots, \vec{w}_{i-1}, \vec{v}_i, \vec{w}_{i+1}, \ldots, \vec{w}_n$ for every value of $i$ between 1 and $k$. Does it follow that $\vec{v}_1, \ldots, \vec{v}_k, \vec{w}_{k+1}, \ldots, \vec{w}_n$ form a basis?