Matrix Theory: homework #2*
Due 11 September 2013, at start of class

1. Prove or give a counterexample for each of following assertions:

   (a) If vectors $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent, whereas vectors $\vec{v}_1, \dots, \vec{v}_n, \vec{w}$ are linearly dependent, then $\vec{w}$ is a linear combination of $\vec{v}_1, \dots, \vec{v}_n$.

   (b) If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are three vectors such that both $\{\vec{v}_1, \vec{v}_2\}$ and $\{\vec{v}_1, \vec{v}_3\}$ are linearly dependent, then $\{\vec{v}_2, \vec{v}_3\}$ is linearly dependent as well.

2. Let V consist of all infinite sequences $a_0, a_1, a_2, a_3, a_4 \dots$ of numbers such that $a_{n+2} = 5a_{n+1} - 6a_n$ for all integers $n \geq 0$.

   (a) Show that V is a vector space;

   (b) Find two sequences, each of which is of the form $a_n = c^n$, that are both in V;

   (c) Show that the two sequences you found above form a basis for V;

   (d) Find a formula for the n'th term of the sequence that starts $1, 4, 14, 46, \dots$.

3. A *magic square* (of order 3) is a 3-by-3 array of numbers such that the sum of every row, every column, and every big diagonal is the same. For example,

$$\begin{bmatrix} 6 & 1 & 8 \\ 7 & 5 & 3 \\ 2 & 9 & 4 \end{bmatrix}$$

   is a magic square. Let V consist of all magic squares endowed with operations of component-wise addition, and component-wise scalar multiplication.

   Show that V is a vector space, and find a basis for V.

*This homework is from `http://www.borisbukh.org/MatrixTheory13/hw2.pdf`.

4. An square matrix A is called *upper triangular* if all entries lying below the diagonal are zero (i.e., $A_{ij} = 0$ whenever $i > j$). Suppose further that the diagonal entries are non-zero (i.e., $A_{ii} \neq 0$ for all i). Show that the columns of A are linearly independent.

5. Do there exist three matrices A, B, C such that $AB = BA$ and $AC = CA$, but $BC \neq CB$?

6. (Bonus problem) Vectors $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent. Vectors $\vec{w}_1, \ldots, \vec{w}_n$ form a basis, and so are vectors $\vec{w}_1, \ldots, \vec{w}_{i-1}, \vec{v}_i, \vec{w}_{i+1}, \ldots, \vec{w}_n$ for every value of i between 1 and k. Does it follow that $\vec{v}_1, \ldots, \vec{v}_k, \vec{w}_{k+1}, \ldots, \vec{w}_n$ form a basis?