Matrix Theory: homework #11*
Due 20 November 2013, at start of class

1. Show that for any operator $T\colon V \to V$ we have $|\det T| = \det|T|$.

2. *(Problem removed from the homework)*

3. Let T be a self-adjoint operator, and let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be its eigenvalues in decreasing order. Let $\vec{v}_1, \ldots, \vec{v}_n$ be the corresponding eigenvectors.

   (a) Show that $\lambda_1 = \max_{\vec{v}}(A\vec{v}, \vec{v})$ where the maximum if taken over all vectors $\vec{v}$ such that $\|\vec{v}\| = 1$.

   (b) Show that $\lambda_2 = \max_{\vec{v} \perp \vec{v}_1}(A\vec{v}, \vec{v})$ where the maximum is taken over all vectors $\vec{v}$ such that $\|\vec{v}\| = 1$ and $\vec{v} \perp \vec{v}_1$.

4. Let V be a finite-dimensional real inner product space, and S and T are self-adjoint linear operators on V such that $ST = TS$. Prove that there exists an orthonormal basis for V consisting of vectors that are eigenvectors of both S and T.

5. Let T be a positive definite operator. Prove that T is invertible, and that T^{-1} is positive definite.

6. Prove that if A is a positive semidefinite matrix, then the singular values of A are equal to the eigenvalues of A.

7. (Bonus problem) Let A and B be two matrices satisfying $A^* = A$ and $B^* = B$. Suppose A is positive definite. Show that there is a matrix T such that $T^*AT = I$ and T^*BT is a diagonal matrix.

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*This homework is from `http://www.borisbukh.org/MatrixTheory13/hw11.pdf`.