Matrix Theory: homework #10*
Due 13 November 2013, at start of class

1. Let $E$ be a subspace of an inner product space $V$, and let $P_E: V \to V$ be the orthogonal projection onto $E$. Find the eigenvalues and eigenspaces of $P_E$. What are algebraic and geometric multiplicities of the eigenvalues?

2. (a) Show that
$$\|x_1 - x_3\|^2 + \|x_2 - x_4\|^2 \leq \|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \|x_3 - x_4\|^2 + \|x_4 - x_1\|^2$$
for any four vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4$ in a real inner product space.

(b) Deduce that for any 4-gon in $\mathbb{R}^n$ there is either an edge that is of length greater than 1 or some diagonal that is of length smaller than $\sqrt{2}$.

(c) (Bonus problem) Prove that in every 6-gon in $\mathbb{R}^n$, either there is an edge of length greater than 1, or one of the great diagonals has length at most 2.

3. Suppose $A$ is an $n$-by-$n$ matrix. Define
$$A_1 = \frac{1}{2}(A + A^*), \quad A_2 = \frac{i}{2i}(A - A^*).$$

(a) Show that $A_1 = A_1^*$, $A_2 = A_2^*$ and $A = A_1 + iA_2$.

(b) Let $A$ be an $n$-by-$n$ matrix. Prove that the representation in (a) is unique. That is, if $A = B_1 + iB_2$, where $B_1 = B_1^*$ and $B_2 = B_2^*$, then $A_1 = B_1$ and $A_2 = B_2$.

(c) Prove that matrices $C = A + A^*$ and $D = AA^*$ satisfy $C = C^*$ and $D = D^*$.

4. Let $T: V \to V$ be a linear operator on a finite-dimensional inner product space.

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*This homework is from http://www.borisbukh.org/MatrixTheory13/hw10.pdf
(a) Prove that $\ker(T^*T) = \ker(T)$. Deduce that $\text{rank}(T^*T) = \text{rank}(T)$.
(b) Prove that $\text{rank}(TT^*) = \text{rank}(T)$.
(c) Is $\text{rank}(TT^*T) = \text{rank}(T)$?

5. Let $T: V \to V$ be an operator on a finite-dimensional inner product space. Show that $T$ and $T^*$ have the same number of distinct eigenvalues.

6. (Bonus problem) $\{\vec{x}_1, \ldots, \vec{x}_n\}$ and $\{\vec{y}_1, \ldots, \vec{y}_n\}$ are two sets of vectors in a finite-dimensional real inner product space $V$ such that $(\vec{x}_i, \vec{x}_j) = (\vec{y}_i, \vec{y}_j)$ for all $i, j$. Prove that there is a unitary operator $T: V \to V$ such that $T\vec{x}_i = \vec{y}_i$ for all $i$. 
