Matrix Theory: homework #10[*]
Due 13 November 2013, at start of class

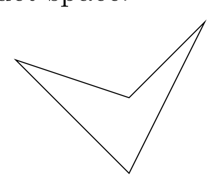
1. Let E be a subspace of an inner product space V, and let $P_E\colon V \to V$ be the orthogonal projection onto E. Find the eigenvalues and eigenspaces of P_E. What are algebraic and geometric multiplicities of the eigenvalues?

2. (a) Show that

$$\|x_1 - x_3\|^2 + \|x_2 - x_4\|^2 \leq \|x_1 - x_2\|^2 + \|x_2 - x_3\|^2 + \|x_3 - x_4\|^2 + \|x_4 - x_1\|^2$$

for any four vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4$ in a real inner product space.

(b) Deduce that for any 4-gon in $\mathbb{R}^n$ there is either an edge that is of length greater than 1 or some diagonal that is of length smaller than $\sqrt{2}$.



4-gon in $\mathbb{R}^2$

(c) (Bonus problem) Prove that in every 6-gon in $\mathbb{R}^n$, either there is an edge of length greater than 1, or one of the great diagonals has length at most 2.

3. Suppose A is an n-by-n matrix. Define

$$A_1 = \tfrac{1}{2}(A + A^*), \qquad A_2 = \tfrac{1}{2i}(A - A^*).$$

(a) Show that $A_1 = A_1^*$, $A_2 = A_2^*$ and $A = A_1 + iA_2$.

(b) Let A be an n-by-n matrix. Prove that the representation in (a) is unique. That is, if $A = B_1 + iB_2$, where $B_1 = B_1^*$ and $B_2 = B_2^*$, then $A_1 = B_1$ and $A_2 = B_2$.

(c) Prove that matrices $C = A + A*$ and $D = AA^*$ satisfy $C = C^*$ and $D = D^*$.

4. Let $T\colon V \to V$ be a linear operator on a finite-dimensional inner product space.

[*]This homework is from `http://www.borisbukh.org/MatrixTheory13/hw10.pdf`.

(a) Prove that $\ker(T^*T) = \ker(T)$. Deduce that $\operatorname{rank}(T^*T) = \operatorname{rank}(T)$.

(b) Prove that $\operatorname{rank}(TT^*) = \operatorname{rank}(T)$.

(c) Is $\operatorname{rank}(TT^*T) = \operatorname{rank}(T)$?

5. Let $T\colon V \to V$ be an operator on a finite-dimensional inner product space. Show that T and T^* have the same number of distinct eigenvalues.

6. (Bonus problem) $\{\vec{x}_1, \ldots, \vec{x}_n\}$ and $\{\vec{y}_1, \ldots, \vec{y}_n\}$ are two sets of vectors in a finite-dimensional real inner product space V such that $(\vec{x}_i, \vec{x}_j) = (\vec{y}_i, \vec{y}_j)$ for all i, j. Prove that there is a unitary operator $T\colon V \to V$ such that $T\vec{x}_i = \vec{y}_i$. for all i.