

Math Studies Algebra: homework #8*

Due 29 October 2017, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in L^AT_EX via e-mail under the same rules as the previous homeworks.

1. Recall that an *initial object* in a category \mathbf{C} is an object $A \in \text{Ob}(\mathbf{C})$ such that for every $B \in \text{Ob}(\mathbf{C})$ there is exactly one morphism from A to B .

Let \mathbf{Grp} be the category of groups with the usual morphisms. Let \mathbf{Grp}^{op} be its opposite category.

- (a) Show that \mathbf{Grp} contains an initial object. What is it?
 - (b) Does \mathbf{Grp}^{op} contain an initial object? If yes, what is it?
2. (a) Let K be a field. Let $f \in K[x]$ be a linear polynomial (that is $\deg f = 1$). Show that the quotient ring $K[x]/(f)$ is isomorphic to K . Show also that $K[x]/(f)$ need not be isomorphic to K if K is an integral domain, but not a field.

(b) For which quadratic polynomials $f \in \mathbb{R}[x]$ is the ring $\mathbb{R}[x]/(f)$ a field?
 3. Problem 26 from Section 7.1 of the textbook (about discrete valuation rings).
 4. Problem 27 from Section 7.1 of the textbook (about discrete valuation rings).
 5. (Ungraded problem; suggested for the students taking Math Studies Analysis) Let R be the ring of continuous functions from $[0, 1]$ to \mathbb{R} with the usual (pointwise) operations of addition and multiplication. Show that each maximal ideal in R is of the form $\{f : f(x_0) = 0\}$ for some x_0 .

*This homework is from <http://www.borisbukh.org/MathStudiesAlgebra1819/hw8.pdf>.