Math Studies Algebra: homework #5*
Due 5 October 2018, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail under the same rules as the previous homeworks.

1. (a) Let p be an odd prime. Show that there is no simple group of order $2p^2$.

   (b) Let p, q be distinct odd primes. Show that there is no simple group of order $2pq$.

2. Let p be a prime, and let $a \in \mathbb{N}$. Show that a group of order p^a contains a normal subgroup of order p^b for every $0 \leq b \leq a$.

3. Show that if n is odd, then the set of n-cycles consists of two conjugacy classes of equal size in A_n.

4. A permutation on $\{1, 2, \dots, n\}$ can be naturally treated as a permutation on $\mathbb{N}$, which is inert on $n+1, n+2, \dots$. This way we have an infinite chain of nested groups $A_2 \leq A_3 \leq A_4 \leq \cdots$. Let $A_\infty = \bigcup_{i \in \mathbb{N}} A_i$ be their union. Show that A_∞ is an infinite simple group.

5. (Bonus problem) Let n be a natural number. Show that a finitely generated group G has only finitely many different subgroups of index n. [Hint: given $H \leq G$, consider an action of G on cosets of H.]

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1819/hw5.pdf`.