# Math Studies Algebra: homework #4*
# Due 28 September 2018, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `lastname_alg_hwnum.tex` and `lastname_alg_hwnum.pdf` respectively. Pictures do not have to be typeset a legible photograph of a hand-drawn picture is acceptable.

1. A group $G$ is *$k$-generated* if there is a set $A \subseteq G$ of size $|A| \leq k$ such that $G = \langle A \rangle$. A group is *finitely generated* if it is $k$-generated for some $k$.
   (a) Show that every finitely generated subgroup of $\mathbb{Q}$ is cyclic.
   (b) Deduce from (a) that $\mathbb{Q}$ is not finitely generated. (For this part, you may assume (a) even if you have not solved it.)
   (c) For each natural number $k$, exhibit an example of a group $G$ that is $k$-generated, but not $(k-1)$-generated.

2. Show that $\mathbb{R}$ does not contain a maximal subgroup.

3. Let $G$ be a group, and let $A \subset G$ be any set. Let $A_t$ be those elements of $G$ that are products of at most $t$ elements from $A$. Say that a $A$ generates the group $G$ in *$s$ steps* if $A_s = G$.
   (a) Let $G$ be a finite group. Show that no two-element set $A$ generates $G$ in $\log_2 |G| - 2$ steps.
   (b) Let $p$ be prime. Show that there is a two-element set $A$ that generates $\mathbb{Z}/p\mathbb{Z}$ in $4\sqrt{p}$ steps. [Note: The number 4 is not the smallest possible here.]
   (c) Let $p$ be prime. Show that there is no two-element set $A$ that generates $\mathbb{Z}/p\mathbb{Z}$ in $\frac{1}{4}\sqrt{p}$ steps. [Note: The number $\frac{1}{4}$ is not the largest possible here.]

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1819/hw4.pdf`.

4. Let $p$ be a prime. Let $G = (\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$. Compute $|\mathrm{Aut}(G)|$.

5. (Bonus problem) True or false: every infinite finitely generated group contains at least two maximal subgroups. Justify your answer.