

Math Studies Algebra: homework #3*

Due 21 September 2018, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in L^AT_EX via e-mail under the same rules as the previous three homeworks.

- Suppose $H \leq G$ is a pair of groups. Show that $\bigcap_{g \in G} gHg^{-1}$ is a normal subgroup of G .
 - Suppose H is a subgroup of G such that $|G : H| < \infty$. Show that there is a normal subgroup $N \trianglelefteq G$ such that $N \leq H$ and $|G : N| < \infty$.
- [Ungraded] Let $H \leq K \leq G$. Prove that $|G : H| = |G : K| \cdot |K : H|$. (Note that G is not necessarily finite.)
- Suppose G is a finite group and $N \triangleleft G$ satisfies $\gcd(|N|, |G : N|) = 1$. Show that N is the unique subgroup of G of order $|N|$.
- Let H and K be subgroups of G . For each x define the HK double coset to be the set
$$HxK = \{h x k : h \in H, k \in K\}.$$
 - Show that, for every x and every y , the double cosets HyK and HxK are either equal or disjoint.
 - Prove that $|HxK| = |K| \cdot |H : H \cap xKx^{-1}|$.
- Let p be a prime. Find the smallest integer n such that S_n contains a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$.

*This homework is from <http://www.borisbukh.org/MathStudiesAlgebra1819/hw3.pdf>.