Math Studies Algebra: homework #12*
Due 3 December 2018, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail under the same rules as the previous homeworks.

1. Let I be an ideal in $F[x_1, \dots, x_n]$ and let G be a Gröbner basis for I. Let $R = F[x_1, \dots, x_n]/I$, and consider R as an vector space over F. Show that the set

$$\{x^a + I : a \in \mathbb{Z}_+^n \text{ and there is no } g \in G \text{ such that } \operatorname{LM}(g) \mid x^a\}$$

is a basis for R as an F-vector space.

2. Let $f_1, \dots, f_n \in F[y_1, \dots, y_m]$ be arbitrary, and define the map $\phi\colon F[x_1, \dots, x_n] \to F[y_1, \dots, y_m]$ by

$$\phi\colon g(x_1, \dots, x_n) \mapsto g(f_1, \dots, f_n).$$

(You may use part (a) in your solution of (b); parts (a)+(b) for (c), etc. even if you did not solve the preceding parts).

   (a) (Ungraded; writing the solution down will help with the next part though) Show that ϕ is a ring homomorphism

   (b) Let $J = (x_1 - f_1, \dots, x_n - f_n)$ be an ideal in $F[x_1, \dots, x_n, y_1, \dots, y_m]$. Show that $\ker \phi = J \cap F[x_1, \dots, x_n]$.

   (c) Explain how, given $f_1, \dots, f_n$, to compute a set of generators for $\ker \phi$.

   (d) Let G be a reduced Gröbner basis for $J = (x_1 - f_1, \dots, x_n - f_n)$ with respect to lexicographic ordering in which the y variables are larger than the x variables. Show that $f \in F[y_1, \dots, y_m]$ is in the image of ϕ if and only if $f \bmod G \in F[x_1, \dots, x_n]$.

3. Enjoy Thanksgiving!

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1819/hw12.pdf`.