Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in \LaTeX via e-mail under the same rules as the previous homeworks.

1. Let $I$ be an ideal in $F[x_1,\ldots,x_n]$ and let $G$ be a Gröbner basis for $I$. Let $R = F[x_1,\ldots,x_n]/I$, and consider $R$ as an vector space over $F$. Show that the set
\[
\{x^a + I : a \in \mathbb{Z}_+^n \text{ and there is no } g \in G \text{ such that } \text{LM}(g) \mid x^a\}\n\]
is a basis for $R$ as an $F$-vector space.

2. Let $f_1,\ldots,f_n \in F[y_1,\ldots,y_m]$ be arbitrary, and define the map $\phi: F[x_1,\ldots,x_n] \to F[y_1,\ldots,y_m]$ by
\[
\phi: g(x_1,\ldots,x_n) \mapsto g(f_1,\ldots,f_n).\n\]
(You may use part (a) in your solution of (b); parts (a)+(b) for (c), etc. even if you did not solve the preceding parts).

(a) (Ungraded; writing the solution down will help with the next part though)
Show that $\phi$ is a ring homomorphism

(b) Let $J = (x_1 - f_1,\ldots,x_n - f_n)$ be an ideal in $F[x_1,\ldots,x_n,y_1,\ldots,y_m]$.
Show that $\ker \phi = J \cap F[x_1,\ldots,x_n]$.

(c) Explain how, given $f_1,\ldots,f_n$, to compute a set of generators for $\ker \phi$.

(d) Let $G$ be a reduced Gröbner basis for $J = (x_1 - f_1,\ldots,x_n - f_n)$ with respect to lexicographic ordering in which the $y$ variables are larger than the $x$ variables. Show that $f \in F[y_1,\ldots,y_m]$ is in the image of $\phi$ if and only if $f \mod G \in F[x_1,\ldots,x_n]$.

3. Enjoy Thanksgiving!

---

*This homework is from [http://www.borisbukh.org/MathStudiesAlgebra1819/hw12.pdf](http://www.borisbukh.org/MathStudiesAlgebra1819/hw12.pdf)