Math Studies Algebra: homework #11*
Due 19 November 2018, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail under the same rules as the previous homeworks.

1. Suppose G_1, G_2 are Gröbner bases for ideals I_1, I_2 respectively. Is $G_1 \cup G_2$ necessarily a Gröbner basis for $I_1 + I_2$?

2. Suppose G is a Gröbner basis, and $f \in F[x_1, \ldots, x_n]$. Show that $f \bmod G$ is independent of the order of polynomials in G.

3. Prove that lex-ordering is indeed a monomial ordering.

4. (Fundamental theorem of symmetric polynomials) For a permutation $\pi \in S_n$ and a monomial $x^a = x_1^{a_1} \cdots x_n^{a_n}$, we define $\pi(x^a) = x_{\pi(1)}^{a_1} \cdots x_{\pi(n)}^{a_n}$. For a polynomial $f = \sum c_a x^a$ we put $\pi(f) = \sum c_a \pi(x^a)$.

   Call $f \in F[x]$ *symmetric* if $\pi(f) = f$ for every $\pi \in S_n$. Let $E_d \subset \mathbb{Z}_+^n$ be all vectors made of d ones and $n-d$ zeros. An *elementary symmetric polynomial of degree* d is $\sigma_d \stackrel{\text{def}}{=} \sum_{a \in E_d} x^a$. For example, $\sigma_2 = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n$.

   Below we use lex-ordering.

   (a) Let $f \in F[x]$ be an arbitrary symmetric polynomial, and suppose that $\mathrm{LM}(f) = x^a$. Show that $a_1 \geq a_2 \geq \cdots \geq a_n$.

   (b) Let $g = \sigma_1^{a_1 - a_2} \sigma_2^{a_2 - a_3} \cdots \sigma_n^{a_n}$, where $\mathrm{LM}(f) = x^a$. Show that $\mathrm{LM}(f) = \mathrm{LM}(g)$.

   (c) Use (b) to show that every symmetric polynomial can be written in he form $h(\sigma_1, \sigma_2, \ldots, \sigma_n)$ for some $h \in F[x]$.

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1819/hw11.pdf`.

5. Suppose that F is a subfield of H, that I is an ideal in $F[x_1, \ldots, x_n]$, and that G is a Gröbner basis for I. Let J be the ideal generated by I in $H[x_1, \ldots, x_n]$. Show that G is also a Gröbner basis for J.