Math Studies Algebra: homework #10*
Due 12 November 2018, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail under the same rules as the previous homeworks.

1. (Bonus problem) Vote!

2. Prove that the rings $F[x,y]/(y^2-x)$ and $F[x,y]/(y^2-x^2)$ are not isomorphic for any field F.

3. Let R be an integral domain, and consider the polynomial ring $R[x_1,x_2,\dots,x_n]$. Recall that a degree of a monomial $ax_1^{d_1}x_2^{d_2}\cdots x_n^{d_n}$ is $d_1+\dots+d_n$, and that a polynomial is homogeneous if all of its monomials are of the same degree.

   (a) Show that a homogeneous polynomial $f \in R[x_1,\dots,x_n]$ satisfies $f(\lambda x_1,\dots,\lambda x_n) = \lambda^k f(x_1,\dots,x_n)$ for all $\lambda \in R$. Is converse true, i.e., does $f(\lambda x_1,\dots,\lambda x_n) = \lambda^k f(x_1,\dots,x_n)$ for all $\lambda \in R$ imply that f is homogeneous?

   (b) An ideal I in $R[x_1,\dots,x_n]$ is *homogeneous ideal* if whenever $p \in I$ then each homogeneous component of p is also in I. Show that an ideal is a homogeneous ideal if and only if it may be generated by homogeneous polynomials. [Use induction on degrees for the "if" direction.]

4. Let F be a field. Show that the ring $F[x]$ contains infinitely many primes. (Consider the cases $|F| < \infty$ and $|F| = \infty$ separately.)

5. Let $a,n \in \mathbb{Z}$ be coprime positive integers. Prove that $(x-a)^n = x^n - a$ in the ring $\mathbb{Z}[x]/(n)$ if and only if n is a prime.

6. Let F be a field, and define $R_1 = F[x_1]$, and $R_i = R_{i-1}[x_i]/(x_i^2 - x_{i-1})$ for $i \geq 2$.

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1819/hw10.pdf`.

(a) Show that R_i is isomorphic to $F[x]$, for each i.

(b) Show that the map $R_{i-1} \to R_i$ induced by the inclusion map $R_{i-1} \to R_{i-1}[x_i]$ is injective. (Thus, we may identify R_i with a subring of R_{i+1}.)

(c) Show that $R = \bigcup_i R_i$ is not isomorphic to $F[x]$.

(d) (Extra credit) Is R a PID?