Math Studies Algebra: homework #1*
Due 7 September 2018, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `lastname_alg_hwnum.tex` and `lastname_alg_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. (a) The operation $\star$ on $\mathbb{Z}$ is defined by $x \star y = x - y$. Is $\mathbb{Z}$ a group under $\star$?

   (b) Find an infinite subset $S \subset \mathbb{Q}$ such that the binary function $\star$ defined by $x \star y = x + y + xy$ turns S into a group under $\star$.

2. Determine all the triples $(a, b, c) \in \mathbb{R}^3$ for which the binary operation $x \star y = ax + by + c$ turns $\mathbb{R}$ into a group.

3. Show that if $x \in G$ is an element of the group G then $\{x^n : n \in \mathbb{Z}\}$ is a subgroup of G (called the *cyclic subgroup* of G generated by x).

4. In this problem, a *symmetry* of a set $X \subset \mathbb{R}^n$ is a bijection of X that preserves distances between pairs of points. Put differently, ϕ is a symmetry if we have $\operatorname{dist}(x, x') = \operatorname{dist}\big(\phi(x), \phi(x')\big)$ for all $x, x' \in X$.

   (a) Show that the set of symmetries of X forms a group under composition.

   (b) What are all the symmetries of $X = \mathbb{Z}$?

   (c) What are all the symmetries of $X = \big\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\big\}$?

5. Show that if $a^2 = e$ for all elements a of a group G, then G is abelian.

6. (Bonus problem) Let G be a non-empty finite set with an associative binary operation, which we denote by the concatenation. Assume that for all $a, b, c \in G$ we have $ab = ac \implies b = c$ and $ba = ca \implies b = c$. Show that G

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1718/hw1.pdf`.

is a group. Give an example showing that the conclusion is false if G is not assumed to be finite.