

# Math Studies Algebra: Axiom of Choice.\*

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## About the Axiom of Choice

The usual foundation of mathematics is set theory. Every mathematical object is a set. For example, earlier in the course we defined a group as a set equipped with a certain kind of a binary operation. The most popular axioms for the set theory are those of Zermelo and Fraenkel. We will not discuss these axioms here, and retain the informal approach.

One of the axioms of set theory, the Axiom of Choice, is less self-evident than the rest. It had been speculated that it might be possible to derive it from the remaining axioms. However, it has been proved that the axiom can neither be proved nor refuted from the remaining axioms.

These notes discuss the Axiom of Choice and two statements that are equivalent to it, Zorn's lemma and the well-ordering principle.

**Axiom of Choice** Informally, the axiom of choice says that it is possible to choose an element from every set. Formally, a *choice function* on a set  $X$  is a function  $f: 2^X \setminus \{\emptyset\} \rightarrow X$  such that  $f(S) \in S$  for every non-empty  $S \subset X$ . The Axiom of Choice asserts that on every set there is a choice function.

**Zorn's Lemma** A *partially ordered set* (poset) is a set whose elements can be compared, but not every pair of elements is comparable. Formally, a partially ordered set is a set  $P$  together with a binary relation  $\preceq$  satisfying

1.  $x \preceq x$ ,
2.  $x \preceq y$  and  $y \preceq z$  imply  $x \preceq z$ ,
3.  $x \preceq y$  and  $y \preceq x$  imply  $x = y$ .

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\*These notes are available from the course webpage, and directly from [http://www.borisbukh.org/MathStudiesAlgebra1718/notes\\_ac.pdf](http://www.borisbukh.org/MathStudiesAlgebra1718/notes_ac.pdf)

If  $x \preceq y$  and  $x \neq y$ , then we write  $x \prec y$  and say that  $x$  is *smaller* than  $y$ . It might happen, for two elements  $x$  and  $y$ , that neither of them is smaller than the other. In that case we say that  $x$  and  $y$  are *incomparable*.

An element  $x$  is called *maximal* if in  $P$  there is no larger element. It is called *maximum* if it is larger than all the other elements. Minimal and minimum elements are defined similarly.

**Example 1:** In  $\mathbb{Z}$  endowed with the usual ordering, every two elements are comparable. However, there are no minimal and no maximal elements.

**Example 2:** The set  $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$  has the minimum element 0, but no maximal elements.

**Example 3:** Consider  $\mathbb{Z}_+ \cup \{\omega\} = \{0, 1, 2, \dots, \omega\}$  where  $\omega$  is a new element that is greater than each element of  $\mathbb{Z}_+$ . In this poset  $\omega$  is the maximum.

**Example 4:** On any set one can define trivial partial order in which no two distinct elements are comparable. In such a poset, every element is both minimal and maximal. There are no maximum and minimal elements (if the poset contains at least two elements).

**Example 5:** If  $X$  is any set, then the relation  $\subseteq$  makes  $2^X = \{S : S \subseteq X\}$  into a poset. Here, the empty set is the minimum, and  $X$  is the maximum.

A *chain* is a set  $C$  all of whose elements are comparable to each other. Note that a subset of chain is also a chain. The Examples 1–3 are chains.

We say that an element  $u$  is an *upper bound* for a chain  $C$  if  $x \preceq u$  for each  $x \in C$ .

Zorn's lemma asserts that if  $P$  is a non-empty poset in which each chain has an upper bound, then  $P$  has a maximal element.

**Well-ordering principle** A poset  $P$  is called *well-ordered* if it is a chain, and every non-empty subset  $S \subset P$  has a minimum. The well-ordering principle asserts that every set can be well-ordered by a suitable relation.

## Equivalence of Axiom of Choice, Zorn's Lemma and the well-ordering principle

**Zorn's lemma implies Axiom of Choice** Let  $X$  be any non-empty set. Aided by Zorn's lemma, we will construct a choice function on  $X$ . Consider pairs  $(Y, f)$  consisting of a subset  $Y \subseteq X$  and a choice function  $f$  on  $Y$ . We introduce a partial order on the set of all such pairs by defining  $(Y, f) \preceq (Y', f')$  whenever  $Y \subseteq Y'$  and  $f = f'|_Y$ .

The poset is non-empty because for every  $x \in X$ , there is an (obvious) partial choice function on  $\{x\}$ . If  $C$  is a chain in this poset, then we can define  $\bar{Y} = \bigcup_{(Y, f) \in C} Y$  and  $\bar{f}(S) = f(S)$  for any  $S$  such that  $f$  is defined on  $S$ . Then  $(\bar{Y}, \bar{f})$  is an upper bound for  $C$ .

Hence, by Zorn's lemma there is some maximal element, which we call  $(f, Y)$ . If  $x \in X \setminus Y$ , then we can extend  $f$  from  $Y$  to  $Y \cup \{x\}$  by defining  $f(S)$  to be equal to  $x$  for any  $S$  containing  $x$ . This contradicts maximality, and so  $X \setminus Y = \emptyset$ , and so  $f$  is a choice function for  $X$ .

**Zorn's lemma implies well-ordering principle** We may assume that the set  $X$  is non-empty, for the empty set is trivially well-ordered.

An *initial segment* of a chain  $C$  is subchain  $C'$  such that  $x \in C$ ,  $y \in C'$  and  $x \prec y$  imply that  $x \in C'$ .

Consider pairs  $(Y, \leq_Y)$ , consisting of a subset  $Y \subseteq X$  and a well-ordering  $\leq_Y$  on  $Y$ . We define a partial order on the set of all such pairs in the similar manner to the preceding proof. Namely,  $(Y, \leq_Y) \preceq (Y', \leq_{Y'})$  whenever  $Y \subseteq Y'$ , the set  $Y$  is an initial segment of  $Y'$  in  $\leq_{Y'}$ , and the two orderings  $\leq_Y$  and  $\leq_{Y'}$  agree on the set  $Y$ .

Since  $X$  is non-empty, the poset is non-empty. Furthermore, if  $C$  is a chain in this poset, we can define  $\bar{Y} = \bigcup_{(Y, \leq_Y) \in C} Y$  and  $x \leq_{\bar{Y}} y$  whenever  $x \leq_Y y$  for some  $(Y, \leq_Y) \in C$ . Then  $\leq_{\bar{Y}}$  is a well-ordering on  $\bar{Y}$ . Indeed, suppose that a set  $S \subseteq \bar{Y}$  is non-empty and  $(Y, \leq_Y) \in C$  is any pair in the chain such that  $S \cap Y \neq \emptyset$ . Let  $u \stackrel{\text{def}}{=} \min_{\leq_Y}(S \cap Y)$ , where the minimum is with respect to  $\leq_Y$ . Then  $u$  is a minimum for  $S$  with respect to  $\leq_{\bar{Y}}$ , for if  $s \in S$  is arbitrary, then either  $s \in Y$  in which case  $u \leq_{\bar{Y}} s$  follows from  $u \leq_Y s$ , or  $s \notin Y$ , in which case  $u \leq_{\bar{Y}} s$  follows from the fact that  $Y$  is an initial segment of  $\bar{Y}$ . Hence, the pair  $(\bar{Y}, \leq_{\bar{Y}})$  is an upper bound for  $C$ .

So, by Zorn's lemma the poset contains a maximal element  $(Y, \leq_Y)$ . If  $Y \neq X$  and  $x \in X \setminus Y$ , then we can extend  $(Y, \leq_Y)$  to a set  $Y \cup \{x\}$  by defining  $x$  to be greater than every element of  $Y$ . This contradicts maximality, and so  $Y = X$ , i.e.,  $X$  can be well-ordered.

**Well-ordering principle implies Axiom of Choice** Suppose  $X$  is a set, and  $\leq$  is a well-ordering of  $X$ . Then  $f(S) = \min S$  defines a choice function on  $X$ .

**Axiom of Choice implies Zorn's Lemma (intuition for the proof)** Let  $P$  be any non-empty poset such that every chain has an upper bound. Assume for contradiction's sake that  $P$  has no maximal element. Pick an element  $x_0$  from  $P$  (using the choice function). Since  $x_0$  is not maximal, there is some  $x_1$  such that  $x_0 \prec x_1$ . Again,  $x_1$  is not maximal, and so there is some  $x_2$  that is greater. We thus obtain a chain  $x_0 \prec x_1 \prec x_2 \prec \dots$ . By the assumption, this chain admits an upper bound  $x_\omega$ . Since  $x_\omega$  is not maximal, there is  $x_{\omega+1}$  that is greater than  $x_\omega$ . Keep on going to obtain the chain  $x_0 \prec x_1 \prec x_2 \prec \dots \prec x_\omega \prec x_{\omega+1} \prec \dots$ . There is an upper bound,  $x_{\omega+\omega}$ , etc. Then, if we believe in magic, then we can argue that this way we can build chains "longer" than  $P$ . The contradiction shows that maximal elements do exist after all. This magic does exist, and is called transfinite induction. Our formal proof below avoids formal discussion of transfinite discussion. Instead it uses a shortcut (suggested to me by Prof. James Cummings).

**Axiom of Choice implies Zorn's Lemma (formal proof)** Let  $P$  be any non-empty poset such that every chain has an upper bound. Assume for contradiction's sake that  $P$  has no maximal element. Let  $f$  be a choice function on  $P$ , and let  $x_0 \stackrel{\text{def}}{=} f(P)$ . If  $C$  is chain, let  $\text{Upp}(C) \stackrel{\text{def}}{=} \{u \notin C : \forall x \in C, x \prec u\}$  be set of all *strict* upper bounds for  $C$ .

**Lemma 1.** *For any chain  $C$ , the set  $\text{Upp}(C)$  is non-empty.*

*Proof.* Let  $u$  be an upper bound for  $C$  (which exists by the assumption on  $P$ ). If  $C$  has no maximum element, then  $u \notin C$ , and so  $u \in \text{Upp}(C)$ . Suppose next that  $C$  contain a maximum element, which we call  $m$ . Since  $P$  has no maximal element, there is  $u$  that is greater than  $m$ . Then  $x \preceq m \prec u$  for each  $x \in C$ , and so  $u \in \text{Upp}(C)$ .  $\square$

For any chain  $C$ , let

$$g(C) \stackrel{\text{def}}{=} f(\text{Upp}(C)).$$

We finally come to the point where we can formalize our informal attempts to build a chain  $x_0, x_1, x_2, \dots, x_\omega, x_{\omega+1}, \dots$ . For purpose of this proof, an *attempt* is a well-ordered set  $A \subset P$  satisfying the following:

1.  $\min A = x_0$ ,
2. For every proper initial segment  $C \subset A$ , we have  $\min A \setminus C = g(C)$ .

**Lemma 2.** *If  $A$  and  $A'$  are two attempts, then either  $A \subseteq A'$  or  $A' \subseteq A$ .*

*Proof.* Suppose the opposite, and let  $z = \min A \setminus A'$  and  $z' = \min A' \setminus A$ . These are well-defined since  $A$  and  $A'$  are well-ordered, respectively.

Since  $z \neq z'$ , we cannot have both  $z \preceq z'$  and  $z' \preceq z$ . Without loss of generality, suppose  $z' \not\preceq z$ . Let  $C = \{x \in A : x \prec z\}$ . From the definition of  $z$  it follows that  $C \subseteq A'$ . It is clear that  $z = \min A \setminus C$ , and so  $z = g(C)$ .

If  $C = A'$ , then  $A' \subset A$ , and we are done. So, suppose that  $C \neq A'$ . If  $z' \preceq x$  for some  $x \in C$ , then transitivity would have implied that  $z' \prec z$ , contrary to our assumption. So, since  $A'$  is chain,  $x \preceq z'$  for every  $x \in C$ . Therefore  $C$  is a proper initial segment of  $A'$ , and so  $g(C) \in A'$ . However,  $g(C) = z \notin A'$ . The contradiction completes the proof.  $\square$

A consequence of the preceding lemma is that union of any set of attempts is an attempt. So, let  $\mathcal{A}$  be the set of all attempts, and put  $A \stackrel{\text{def}}{=} \bigcup_{A \in \mathcal{A}} A$ . Then  $A$  is an attempt. However,  $A \cup \{g(A)\}$  is an attempt that contains  $A$ . The contradiction shows  $P$  does have a maximal element after all.