Math Studies Algebra: homework #9*
Due 2 November 2016, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail under the same rules as the previous homeworks.

1. A ring R is called *Boolean* if $a^2 = a$ for all $a \in R$. Show that every Boolean ring is commutative.

2. Suppose $R_1, R_2, \dots$ is an infinite sequence of rings, and we have ring homomorphisms $\phi_i \colon R_{i+1} \to R_i$ for each i. Let R be the set whose elements are sequences $(x_1, x_2, \dots)$ with $x_i \in R_i$ satisfying $\phi_i(x_{i+1}) = x_i$. Define addition and multiplication componentwise.

   (a) Show that with these operations, R is a ring.

   (b) Let $N \geq 2$ be an integer. Show that the map $\phi_i \colon Z/N^{i+1}\mathbb{Z} \to Z/N^i\mathbb{Z}$ given by $\phi_i(x) \stackrel{\text{def}}{=} x + N^i\mathbb{Z}$ is a ring homomorphism.

   (c) Let $N = 7$, and $R_i = \mathbb{Z}/7^i\mathbb{Z}$, and let $\phi_i \colon \mathbb{Z}/7^{i+1}\mathbb{Z} \to \mathbb{Z}/7^i\mathbb{Z}$ be as above. Define R as in part (a). Show that the equation $x^2 = 2$ has a solution in R. (Here $2 = (2, 2, 2, \dots)$.)

3. The *center* of a ring R is $\{z \in R : zr = rz \text{ for all } r \in R\}$.

   (a) Show that the center of R is a subring of R.

   (b) Let R be a commutative ring with the identity, and let G be a group. Find the center of the group ring RG.

4. (Bonus problem) Show that if a ring R is infinite, then the number of zero divisors in R is either zero or infinite.

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1617/hw9.pdf`.