Math Studies Algebra: homework #7*
Due 19 October 2016, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail under the same rules as the previous homeworks.

1. (a) Let p be an odd prime. Show that there is no simple group of order $2p^2$.

   (b) Let p, q be distinct odd primes. Show that there is no simple group of order $2pq$.

2. Let p be a prime, and let $a \in \mathbb{N}$. Show that a group of order p^a contains a normal subgroup of order p^b for every $0 \leq b \leq a$.

3. (a) Suppose that a finite group G acts on a finite set A. For $g \in G$ let $\mathrm{Fix}(g) = \{a \in A : g \cdot a = a\}$ be the set of its fixed points. Show that the number of orbits is
$$\frac{1}{|G|} \sum |\mathrm{Fix}(g)|.$$

   (b) Let n be an odd prime. In how many ways can one make a round necklace out of n blue and n red beads? [Two necklaces are considered same if one can be obtained from another by a rigid motion. Your answer might involve binomial coefficients.]

4. Prove that in the group $G = \langle r, s \mid r^n = s^2 = srsr = 1 \rangle$ the elements r^i, sr^i for $i = 0, 1, \ldots, n-1$ are distinct. (Note that in our sketch of $G \cong D_{2n}$ in class, we did not prove this. It is your job now.)

5. (Bonus problem) Let n be a natural number. Show that a finitely generated group G has only finitely many different subgroups of index n. [Hint: given $H \leq G$, consider an action of G on cosets of H.]

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1617/hw7.pdf`.