Math Studies Algebra: homework #6*
Due 12 October 2016, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail under the same rules as the previous homeworks.

1. Find all the finite groups that have exactly two conjugacy classes.

2. An element of the group is *gloomy* if it does not commute with any other element except for the identity and itself. Show that, in a finite group, either no element is gloomy or exactly half of the elements are. [The terminology in this problem is non-standard.]

3. (a) For a group G, let $\phi\colon G \to \mathrm{Inn}(G)$ be the homomorphism given by $g \mapsto (h \mapsto ghg^{-1})$. Give an example of a nontrivial group G for which ϕ is an isomorphism between G and $\mathrm{Inn}(G)$.

   (b) Give an example of a group G which satisfies both $|\mathrm{Inn}(G)| < \infty$ and $|\mathrm{Aut}(G)| = \infty$.

4. Show that if n is odd, then the set of n-cycles consists of two conjugacy classes of equal size in A_n.

5. Let p be a prime. Prove that $a^{p-1} \equiv 1 \pmod{p}$ whenever $\gcd(a, p) = 1$ by looking at the group $(\mathbb{Z}/p\mathbb{Z})^*$.

6. A permutation on $\{1, 2, \ldots, n\}$ can be naturally treated as a permutation on $\mathbb{N}$, which is inert on $n+1, n+2, \ldots$. This way we have an infinite chain of nested groups $A_2 \leq A_3 \leq A_4 \leq \cdots$. Let $A_\infty = \bigcup_{i\in\mathbb{N}} A_i$ be their union. Show that A_∞ is an infinite simple group.

7. (Bonus problem) Let $[0, 1] \subset \mathbb{R}$ be the usual unit interval. Consider the group G of all the invertible continuous function $f\colon [0, 1] \to [0, 1]$ under the operation of composition. Is G simple?

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1617/hw6.pdf`.