Math Studies Algebra: homework #5*
Due 5 October 2015, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail under the same rules as the previous three homeworks.

1. (a) Let H and K be subgroups of G. For each x define the HK *double coset* to be the set
$$HxK = \{hxk : h \in H, k \in K\}.$$
Show that, for every x and every y, the double cosets HyK and HxK are either equal or disjoint.
   (b) Prove that $|HxK| = |K| \cdot |H : H \cap xKx^{-1}|$.

2. (a) Show that S_n is 2-generated, i.e., there exist $\pi_1, \pi_2 \in S_n$ such that $S_n = \langle \pi_1, \pi_2 \rangle$. (Hint: one of π_1, π_2 can be chosen to be a cycle.)
   (b) (Bonus problem) Suppose that n is odd. Show that A_n is 2-generated.

3. Let p be a prime. Find the smallest integer n such that S_n contains a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^2$.

4. Let G be a finite group, and let $\pi\colon G \to S_G$ be the left regular representation. Prove that if x is an element of order n and $|G| = nm$, then $\pi(x)$ is a product of m n-cycles. Deduce that $\pi(x)$ is an odd permutation if and only if $|x|$ is even and $|G|/|x|$ is odd.

5. Let $n \geq 5$. Is there a homomorphism from A_n to itself whose image is neither $\{\text{id}\}$ nor A_n?

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1617/hw5.pdf`.