Math Studies Algebra: homework #4*
Due 30 September 2015, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail under the same rules as the previous three homeworks.

1. (a) Suppose $H \leq G$ is a pair of groups. Show that $\bigcap_{g \in G} gHg^{-1}$ is a normal subgroup of G.

   (b) Suppose H is a subgroup of G such that $|G : H| < \infty$. Show that there is a normal subgroup $N \trianglelefteq G$ such that $N \leq H$ and $|G : N| < \infty$.

2. Let $H \leq K \leq G$. Prove that $|G : H| = |G : K| \cdot |K : H|$. (Note that G is not necessarily finite.)

3. Suppose G is a finite group and $N \triangleleft G$ satisfies $\gcd(|N|, |G : N|) = 1$. Show that N is the unique subgroup of G of order $|N|$.

4. Prove that A_n contains a subgroup isomorphic to S_{n-2}.

5. Prove that two cycles π_1, π_2 in S_n commute if and only if either they are disjoint, or one of them is a power of another.

6. Let G be any group. Let N be the group generated by all the elements of the form $aba^{-1}b^{-1}$ for some $a, b \in G$.

   (a) Show that N is normal.

   (b) Show that G/N is abelian. (For this part, you can assume that N is normal even if you have not solved part (a).)

   (c) (Bonus problem) Show that if $\phi \colon G \to H$ is a homomorphism into an abelian group H, then ϕ is of the form $\phi = \psi \circ \pi$ for some homomorphism $\psi \colon G/N \to H$. Here, $\pi \colon G \to G/N$ is the natural projection.

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1516/hw4.pdf`.