Math Studies Algebra: homework #3*
Due 23 September 2015, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `lastname_alg_hwnum.tex` and `lastname_alg_hwnum.pdf` respectively. Pictures do not have to be typeset a legible photograph of a hand-drawn picture is acceptable.

1. Let $n \geq 3$. Find all the normal subgroups of the dihedral group D_{2n}.

2. Let G be an abelian group containing elements a and b of orders m and n respectively. Show that G contains an element whose order is $\mathrm{lcm}(m, n)$. [If stuck, try the case $\gcd(m, n) = 1$ first.]

3. (a) Prove that a group N of G is normal if and only if $gNg^{-1} \subseteq N$ for *all* g.
   (b) Let $G = GL_2(\mathbb{Q})$, let $N = \{\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) : x \in \mathbb{Z}\}$, and let $g = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Show that $gNg^{-1} \subseteq N$, but g does *not* normalize N.

4. A group G is *k-generated* if there is a set $A \subseteq G$ of size $|A| \leq k$ such that $G = \langle A \rangle$. A group is *finitely generated* if it is k-generated for some k.
   (a) Show that every finitely generated subgroup of $\mathbb{Q}$ is cyclic.
   (b) Deduce from (a) that $\mathbb{Q}$ is not finitely generated. (For this part, you may assume (a) even if you have not solved it.)
   (c) For each natural number k, exhibit an example of a group G that is k-generated, but not $(k-1)$-generated.

5. Let p be a prime. Let $G = (\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$. Compute $|\mathrm{Aut}(G)|$.

6. (Bonus problem) True or false: every infinite finitely generated group contains at least two maximal subgroups. Justify your answer.

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1516/hw3.pdf`.