Math Studies Algebra: homework #21*
Due 6 April 2016, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail under the same rules as the previous homeworks.

1. For a field F let $\mathcal{P}_n(F)$ be the set of all monic irreducible polynomials of degree n in $F[x]$. Let p be a prime.

   (a) Let $\mathbb{F}_p$ be the finite field with p elements. Prove that
   $$\prod_{n|m} \prod_{f \in \mathcal{P}_n(\mathbb{F}_p)} f = x^{p^m} - x,$$
   where the first product extends over all integers n dividing m.

   (b) Suppose $q = p^t$ for some $t \in \mathbb{N}$. Compute a closed form for
   $$\prod_{n|m} \prod_{f \in \mathcal{P}_n(\mathbb{F}_q)} f.$$

2. Let $p_1, p_2, \ldots, p_t$ be distinct prime numbers. Let $F = \mathbb{Q}[\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_t}]$. Describe all elements of $\operatorname{Aut}(F/\mathbb{Q})$. [Note: it is not enough to find an abstract group that is isomorphic to $\operatorname{Aut}(F/\mathbb{Q})$.]

3. (a) Let $f \in \mathbb{C}(x)$. Show that if $\mathbb{C}(f) = \mathbb{C}(x)$ then f must be the form $(ax+b)/(cx+d)$ for some $a, b, c, d \in \mathbb{C}$.

   (b) Use (a) to compute $\operatorname{Aut}\big(\mathbb{C}(x)/\mathbb{C}\big)$.

4. Let F be a field. Compute the fixed field of automorphism $x \mapsto x+1$ of $F(x)$.

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6. Give an example of an extension K/F of degree 2 such that K is not of the form $F(\sqrt{D})$ for some $D \in F$. [Example after Corollary 13 in Section 13.2 is relevant.]

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1516/hw21.pdf`.