# Math Studies Algebra: homework #20*
# Due 30 March 2016, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail under the same rules as the previous homeworks.

1. Let $\alpha$ be algebraic over $\mathbb{Q}$. Show that there is an $n \in \mathbb{Z}$ such that $n\alpha$ is an algebraic integer.

2. Let $p$ be prime. Let $F = \mathbb{Z}/(p)$ be a field with $p$ elements. Consider the field $F(x, y)$. Let $a, b \in \overline{F(x, y)}$ be elements satisfying $a^p - x = 0$ and $b^p - y = 0$.
   (a) Show that for every $w \in F(a, b)$ we have $w^p \in F(x, y)$.
   (b) Show that the extension $F(a, b)/F(x, y)$ is finite, but not simple.

3. Exercise 15 in Section 13.2

4. (a) Let $e = 2.71\ldots$ be the Euler's number. Let $n \geq 1$ be an integer. Prove that the distance from $n!e$ to the nearest integer lies in the interval $(0, 1/n)$. Deduce that $e$ is irrational.
   (b) Use a similar argument to show that $e$ satisfies no non-trivial quadratic equation with rational coefficients.

5. Let $F$ be a field of positive characteristic $p$. Suppose $a \in F$ is not a $p$'th power. Show that, for each $t \in \mathbb{N}$, polynomial $x^{p^t} - a$ is irreducible in $\mathbb{F}[x]$.

6. Let $F$ be a field with finitely many elements. Prove that $|F|$ is a power of a prime.

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1516/hw20.pdf`.