Math Studies Algebra: homework #19*
Due 23 March 2016, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail under the same rules as the previous homeworks. Below 'representation' means a 'finite-dimensional representation of a finite group over $\mathbb{C}$'.

1. Let G be a group, let and $[G, G]$ be its commutator subgroup. Show that the number of degree-one representations of G is equal to the number of irreducible representations of $G/[G, G]$.

2. Let χ be a character of an irreducible representation of a group G. Prove that

$$\sum_{g \in G} \chi(g) = \begin{cases} |G| & \text{if } \chi = \chi_{\text{id}} \\ 0 & \text{if } \chi \neq \chi_{\text{id}}; \end{cases}$$

   where χ_{id} is the trivial degree-one character.

3. (a) For $f \colon G \to \mathbb{C}$ and an irreducible representation ρ on vector space V, let

$$\hat{f}(\rho) \stackrel{\text{def}}{=} \sum_{g \in G} f(g)\rho(g).$$

   Note that $\hat{f} \in GL(V)$.

   Let $\rho_1, \rho_2, \ldots, \rho_h$ be the irreducible representations of G, and let $n_1, n_2, \ldots, n_h$ be their degrees. Show that

$$f(g) = \frac{1}{|G|} \sum_{i=1}^{h} n_i \operatorname{tr}\big(\rho_i(g^{-1})\hat{f}(\rho_i)\big)$$

   (The function $\rho \mapsto \hat{f}(\rho)$ is called *Fourier transform* of f.)

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1516/hw19.pdf`.

(b) For two functions $f, f' \in G \to \mathbb{C}$, define their convolution $f * f' \colon G \to \mathbb{C}$ by

$$f * f'(h) \overset{\text{def}}{=} \sum_{gg'=h} f(g) f'(g'),$$

where the summation is over all pairs $(g, g') \in G^2$ such that $gg' = h$. Show that $\widehat{f * f'}(\rho) = \hat{f}(\rho)\hat{f}'(\rho)$.

4. True or false? Justify.

   (a) For two irreducible monic polynomials $p, q \in \mathbb{Q}[x]$, the fields $F[x]/(p)$ and $F[x]/(q)$ are isomorphic if and only if $p(x) = q(x+c)$ for some $c \in \mathbb{Q}$.

   (b) No field is isomorphic to its proper subfield.

5. Let $p_1, p_2, \ldots, p_k$ be distinct prime numbers. Show that the field $\mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_k})$ has degree 2^k over $\mathbb{Q}$.