Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in L\TeX{} via e-mail under the same rules as the previous homeworks. Below ‘representation’ means a ‘finite-dimensional representation of a finite group over \( \mathbb{C} \).

1. Let \( G \) be a group, let and \([G, G]\) be its commutator subgroup. Show that the number of degree-one representations of \( G \) is equal to the number of irreducible representations of \( G/[G, G] \).

2. Let \( \chi \) be a character of an irreducible representation of a group \( G \). Prove that

\[
\sum_{g \in G} \chi(g) = \begin{cases} |G| & \text{if } \chi = \chi_{\text{id}} \\ 0 & \text{if } \chi \neq \chi_{\text{id}} \end{cases}
\]

where \( \chi_{\text{id}} \) is the trivial degree-one character.

3. (a) For \( f : G \to \mathbb{C} \) and an irreducible representation \( \rho \) on vector space \( V \), let

\[
\hat{f}(\rho) \overset{\text{def}}{=} \sum_{g \in G} f(g) \rho(g).
\]

Note that \( \hat{f} \in GL(V) \).

Let \( \rho_1, \rho_2, \ldots, \rho_h \) be the irreducible representations of \( G \), and let \( n_1, n_2, \ldots, n_h \) be their degrees. Show that

\[
f(g) = \frac{1}{|G|} \sum_{i=1}^{h} n_i \text{tr}(\rho_i(g^{-1}) \hat{f}(\rho_i))
\]

(The function \( \rho \mapsto \hat{f}(\rho) \) is called Fourier transform of \( f \).)

*This homework is from http://www.borisbukh.org/MathStudiesAlgebra1516/hw19.pdf
(b) For two functions $f, f' \in G \rightarrow \mathbb{C}$, define their convolution $f * f' : G \rightarrow \mathbb{C}$ by

$$f * f'(h) \overset{\text{def}}{=} \sum_{gg' = h} f(g) f'(g'),$$

where the summation is over all pairs $(g, g') \in G^2$ such that $gg' = h$.

Show that $\hat{f} \hat{f}'(\rho) = \hat{f}(\rho) \hat{f}'(\rho)$.

4. True or false? Justify.

(a) For two irreducible monic polynomials $p, q \in \mathbb{Q}[x]$, the fields $F[x]/(p)$ and $F[x]/(q)$ are isomorphic if and only if $p(x) = q(x+c)$ for some $c \in \mathbb{Q}$.

(b) No field is isomorphic to its proper subfield.

5. Let $p_1, p_2, \ldots, p_k$ be distinct prime numbers. Show that the field $\mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_k})$ has degree $2^k$ over $\mathbb{Q}$. 