Math Studies Algebra: homework #17*
Due 24 February 2016, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail under the same rules as the previous homeworks. In this homework, letter F always denotes a field, and letter G always denotes a finite group.

1. Let V be a (possibly infinite dimensional) FG-module. Prove that for each $v \in V$ there is an FG-submodule of V containing v of dimension $\leq |G|$ (as a vector space over F).

2. Suppose $|G|$ is divisible by $\operatorname{char} F$. Let R be the regular representation of G over F. Let $N \stackrel{\text{def}}{=} \{\sum_{g \in G} \alpha g : \alpha \in F\}$. Show that N is a subrepresentation of R, but there is no subrepresentation M of R such that $M \oplus N = R$.

3. Suppose V is vector space over $\mathbb{C}$, and $\rho\colon G \to \mathrm{GL}(V)$ is a representation of G. Let V^* be the vector space dual of V. For $x^* \in V^*$ and $x \in V$, denote by $\langle x, x^* \rangle$ the value of x^* on x. Show that there exists a unique representation $\rho^*\colon G \to \mathrm{GL}(V^*)$ such that
$$\langle x, x^* \rangle = \langle \rho(g)x, \rho^*(g)x^* \rangle \qquad \text{for all } x \in V,\ x^* \in V^*,\ g \in G.$$

4. Suppose F' is a field containing F. Let V be a vector space over F. Let $V' = V \otimes_F F'$.
   (a) Show that the map $\tau\colon\ \mathrm{GL}(V) \to \mathrm{GL}(V')$ given by $T \mapsto T \otimes 1$ is a group homomorphism.
   (b) Give an example of a representation $\rho\colon G \to GL(V)$ that is irreducible, but such that the representation $\rho'\colon G \to GL(V')$ given by $\rho' = \tau \circ \rho$ is reducible. [Hint: this can be done with G being cyclic.]

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1516/hw17.pdf`.