Math Studies Algebra: homework #16*
Due 17 February 2016, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail under the same rules as the previous homeworks.

1. Exercise 3 in section 12.1. [Note that in this exercise ring R is not assumed to be principal. Also note that the hint given in the book is an overkill; you may consider doing explicit Gaussian elimination instead.]

2. Suppose R is a principal ideal domain, and M and N are R-modules of finite ranks m and n respectively. For each of the following two statements give a proof or a counterexample.

   (a) Rank of $M \otimes_R N$ is at least mn.

   (b) Rank of $M \otimes_R N$ is at most mn.

3. Let R be an integral domain. Show that there exists a torsion-free R-module M of rank 1 such that for every torsion-free R-module N of rank 1 there is a submodule N' of M which is isomorphic to N.

4. Exercise 18 in section 12.2

5. Exercise 20 in section 12.2

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1516/hw16.pdf`.