Math Studies Algebra: homework #15*
Due 10 February 2016, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail under the same rules as the previous homeworks.

1. Let R be a ring with 1.
   (a) Show that if N and N' are Noetherian R-modules, then $N \oplus N'$ is a Noetherian R-module.
   (b) Show if an infinite direct sum $N_1 \oplus N_2 \oplus \cdots$ of Noetherian R-modules is a Noetherian R-module if and only if $N_i = 0$ for all except finitely many i's.
2. (a) Let R be a PID, and let M be a free R-module of finite rank. Suppose N be a R-submodule of M. Let $b \in M$ be arbitrary. Show that $b \notin N$ if and only if there exists an ideal I in R and a homomorphism $\phi \in \mathrm{Hom}_R(M, R/I)$ such that $\phi(N) = 0$ and $\phi(b) \neq 0$.
   (b) Suppose $a_{i,j}$ and b_i are rational numbers. Prove that a system of equations
$$\begin{aligned} a_{1,1}x_1 + \cdots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + \cdots + a_{2,n}x_n &= b_2 \\ &\vdots \\ a_{m,1}x_1 + \cdots + a_{m,n}x_n &= b_m \end{aligned}$$
   admits no integer solution $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ if and only if there exist $c_1, \ldots, c_n \in \mathbb{Q}$ such that $\sum_i c_i a_{i,j} \in \mathbb{Z}$ for all j, but $\sum_i c_i b_i \notin \mathbb{Z}$. (You may use part (a) even if you did not solve it.)

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1516/hw15.pdf`.

3. Show that an ideal in a ring R is a free R-module if and only if it is principal. Deduce that if R is not a principal ideal domain, then a submodule of a free R-module need not be free.

4. (a) Let R be a PID. Suppose M is a free R-module, which is not necessarily finitely generated. Assume that an element $x \in M$ and a R-submodule M' of M satisfy $M = Rx \oplus M'$. Show that M' is free.

   (b) (Extra credit; I do not know the answer) Does the part (a) hold if R is an arbitrary ring with 1?

5. Give an example of a principal ideal domain R and a countably generated R-module M that is *not* isomorphic to a direct sum of modules of the form R/I.