Math Studies Algebra: homework #14*
Due 3 February 2016, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail under the same rules as the previous homeworks.

1. The purpose of this exercise is to compare the "tensor product of homomorphisms" (as defined in Theorem 13 on p. 370) with "tensor product of Homs".

   Let R be a commutative ring with 1. Suppose M, M', N, N' are R-modules. Treat $\mathrm{Hom}(M, N)$ and $\mathrm{Hom}(M', N')$ as R-modules.

   For $\phi \in \mathrm{Hom}(M, N)$ and $\psi \in \mathrm{Hom}(M', N')$ we defined $\phi \otimes \psi$ to be a certain element of $\mathrm{Hom}(M \otimes_R M', N \otimes_R N')$. Show that the resulting map $\mathrm{Hom}(M, N) \times \mathrm{Hom}(M', N') \to \mathrm{Hom}(M \otimes_R M', N \otimes_R N')$ is R-bilinear. (So, our definition agrees with the definition of $\mathrm{Hom}(M, N) \otimes_R \mathrm{Hom}(M', N')$ when R is commutative).

2. Exercise 8 in section 11.5

3. Exercise 13 in section 11.5

4. Let V be a vector space over a field F. Let $V^* \stackrel{\text{def}}{=} \mathrm{Hom}_F(V, F)$ be the dual space of V. Let $\mathrm{End}(V) \stackrel{\text{def}}{=} \mathrm{Hom}_F(V, V)$ be the endomorphism ring of V.

   (a) Prove that the map $\Phi\colon V \otimes_F V^* \to \mathrm{End}(V)$ defined on the simple tensors by $v \otimes w \mapsto (u \mapsto w(u)v)$ (and extended by linearity on $V \otimes_F V^*$) is well-defined.

   (b) Show that $\ker \Phi = 0$.

   (c) Prove that if V is a finite-dimensional vector space, then Φ is an isomorphism.

*This homework is from http://www.borisbukh.org/MathStudiesAlgebra1516/hw14.pdf.

(d) Let $\operatorname{tr}\colon V \otimes_F V^* \to F$ be given by $\operatorname{tr}(v \otimes_F w) = w(v)$. By choosing coordinates, show that if V is finite-dimensional, then tr is the trace on $\operatorname{End}(V) \cong M_n(F)$.

(e) (Extra credit) Use the preceding item to show that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

5. Let A be and B be square matrices (over a field). Let $A \otimes B$ be their tensor product.

   (a) Show that $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A)\operatorname{tr}(B)$.

   (b) Compute $\det(A \otimes B)$ in terms of $\det(A)$, $\det(B)$ and sizes of A and B.