Math Studies Algebra: homework #13*
Due 27 January 2015, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail under the same rules as the previous homeworks.

1. Let R and S be rings with 1. Let M, N be an R-module and an S-module respectively. Define an action of $R \oplus S$ on $M \oplus N$ by $(r,s)(m,n) = (rm, sn)$ for all $(r,s) \in R \times S$ and all $(m,n) \in M \times N$.

   (a) Show that $M \oplus N$ with this action is an $R \oplus S$-module.

   (b) Show that every $R \oplus S$-module is isomorphic to an $R \oplus S$-module that arises in this manner.

2. Let R be a subring of S, which in turn is a subring of T. Suppose that R, S, T all contain the identity, and that $1_R = 1_S = 1_T$. Let N be a S-module. Note that N is also an R-module (by restriction). Must the T-modules $T \otimes_R N$ and $T \otimes_S N$ be necessarily isomorphic? Justify your answer.

3. Let R be a commutative ring with 1. Let M, N be R-modules, with M being finitely generated (as an R-module). Let $\{M_i\}_{i \in \mathcal{I}}$ and $\{N_i\}_{i \in \mathcal{I}}$ be two families of R-modules.

   (a) Show that the R-module $\operatorname{Hom}_R(R, R)$ is isomorphic to R.

   (b) Show that $\operatorname{Hom}_R(M, \bigoplus_{i \in \mathcal{I}} N_i) \cong \bigoplus_{i \in \mathcal{I}} \operatorname{Hom}_R(M, N_i)$.

   (c) Show that $\operatorname{Hom}_R(\bigoplus_{i \in \mathcal{I}} M_i, N) \cong \prod_{i \in \mathcal{I}} \operatorname{Hom}_R(M_i, N)$.

4. A R-module M is a *torsion* module if for every nonzero $m \in M$ there is a non-zero $r \in R$ such that $rm = 0$.

   (a) Show that every finite abelian group is a torsion $\mathbb{Z}$-module.

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1516/hw13.pdf`.

(b) Give an example of an infinite abelian group that is a torsion $\mathbb{Z}$-module.

(c) Let R be a commutative ring with 1. Prove that R is a field if and only if the only torsion R-module is the zero module.

5. In the following, "compute" means "give an explicit description not involving tensor products":

   (a) Compute $\mathbb{Z}/(p_1^{e_1}) \otimes_{\mathbb{Z}} \mathbb{Z}/(p_2^{e_2})$ where p_1, p_2 are (possibly equal) primes and $e_1, e_2 \in \mathbb{Z}_+$

   (b) Compute $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ where p is a prime.

   (c) Let p' be a prime. Let $R \subset \mathbb{Q}$ be the group (under addition) consisting all rational numbers whose denominators are relatively prime to p'. Compute $R \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ where p is a prime.