# Math Studies Algebra: homework #11*
# Due 20 November 2015, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail under the same rules as the previous homeworks.

1. Recall that an *initial object* in a category $\mathbf{C}$ is an object $A \in \mathrm{Ob}(\mathbf{C})$ such that for every $B \in \mathrm{Ob}(\mathbf{C})$ there is exactly one morphism from $A$ to $B$.

   Let $\mathbf{Grp}$ be the category of groups with the usual morphisms. Let $\mathbf{Grp}^{op}$ be its opposite category.

   (a) Show that $\mathbf{Grp}$ contains an initial object. What is it?

   (b) Does $\mathbf{Grp}^{op}$ contain an initial object? If yes, what is it?

2. Let $R$ be a commutative ring with the identity, and $a \in R$ arbitrary. Let $m, n$ be positive integers and $d = \gcd(m, n)$. Show that $(a^n - 1, a^m - 1) = (a^d - 1)$.

3. Show that a principal ideal domain $R$ is a discrete valuation ring only if $R$ contains a unique maximal ideal.

4. Let $R$ be a principal ideal domain. Let $I, J$ be ideals in $R$. Show that $I \cap J = IJ$ if and only if $I$ and $J$ are comaximal. Give an example showing that the result is false is $R$ is an integral domain, but not a principal ideal domain.

5. Let $R$ be a principal ideal domain, and let $D$ be a multiplicative set. Show that $R[D^{-1}]$ is a principal ideal domain.

6. What are the units in $\mathbb{Z}[i]$?

7. (Removed from the homework)

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1516/hw11.pdf`.

8. (Bonus problem) Show that the only automorphism of the ring $\mathbb{R}$ is the identity map.