Math Studies Algebra: homework #1*
Due 9 September 2015, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `lastname_alg_hwnum.tex` and `lastname_alg_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. (a) The operation $\star$ on $\mathbb{Z}$ is defined by $x \star y = x - y$. Is Z a group under $\star$?

   (b) Find an infinite subset $S \subset \mathbb{Q}$ such that the binary function $\star$ defined by $x \star y = x + y + xy$ turns S into a group under $\star$.

2. Determine all the triples $(a, b, c) \in \mathbb{R}^3$ for which the binary operation $x \star y = ax + by + c$ turns $\mathbb{R}$ into a group.

3. Show that if $x \in G$ is an element of the group G then $\{x^n : n \in \mathbb{Z}\}$ is a subgroup of G (called the *cyclic subgroup* of G generated by x).

4. Let p be a prime. Show that an element has order p in S_n if and only if its cycle decomposition is a product of commuting p-cycles. Show by an explicit example that this need not be the case if p is not prime.

5. Show that if $a^2 = e$ for all elements a of a group G, then G is abelian.

6. (Bonus problem) Let G be a non-empty finite set with an associative binary operation, which we denote by the concatenation. Assume that for all $a, b, c \in G$ we have $ab = ac \implies b = c$ and $ba = ca \implies b = c$. Show that G is a group. Give an example showing that the conclusion is false if G is not assumed to be finite.

*This homework is from `http://www.borisbukh.org/MathStudiesAlgebra1516/hw1.pdf`.