Geometric combinatorics: supplementary notes 2*

If $P \subset \mathbb{R}^d$ is a finite point set, then $P' \subset P$ is a weak ε-net for P with respect to the convex sets if for every convex set $C \subset \mathbb{R}^d$ containing at least $\varepsilon|P|$ points of P, we have $P' \cap C \neq \emptyset$.

Theorem 1 (Weak ϵ-net theorem). *If $P \subset \mathbb{R}^d$ is finite, then there is a weak $(1/r)$-net P' for P with respect to convex sets of size $|P'| \leq f(r, d)$.*

Lemma 2. *If $P \subset \mathbb{R}^d$ is an n-point set, then there is a hyperplane h that partitions P into two sets, each of size at least $n/2 - 1$.*

Proof. For a generic direction v, every hyperplane with normal v contains at most one point of P. Pick such a direction v, and a hyperplane h normal to v so that P is on one side of h. Slide h towards P in direction v. As the hyperplane slides, the number of points of P in the two parts changes by at most one at a time. □

Proof of the weak ϵ-net theorem. Proof is by induction on d and $\lceil \log_{4/3} r \rceil$. The base case $d = 1$ is easy, whereas the base case $r = 1$ is trivial. Thus, suppose $d \geq 2$. Let h be a hyperplane separating P_1 and P_2 into two sets of size at least $n/2 - 1$ each as above. Let P_1' and P_2' be weak $(4/3r)$-nets for P_1 and P_2 respectively. Note that if C is a convex set satisfying $C \cap P_1' = \emptyset$ and $|C \cap P| \geq (1/r)n$, then

$$|C \cap P_2| \geq |C \cap P| - (4/3r)|P_1| \geq (1/r)n - (4/3r)(n/2 + 1) \geq (1/3r)n - 2.$$

Similarly, $C \cap P_2' = \emptyset$ and $|C \cap P| \geq (1/r)n$ imply $|C \cap P_1| \geq (1/3r)n - 2$. For points $p_1, p_2 \in \mathbb{R}^d$ denote by $[a, b]$ the line segment connecting p_1 and p_2. Let $\mathcal{F} = \{[p_1, p_2] : p_1 \in P_1,\ p_2 \in P_2\}$, and $\tilde{\mathcal{P}} = \{s \cap h : s \in \mathcal{F}\}$. Note that $C \cap (P_1' \cup P_2') = \emptyset$ and $|C \cap P| \geq (1/r)n$ together imply that C contains at least $((1/3r)n - 2)^2$ line segments from $\mathcal{F}$, and hence the same number of points of $\tilde{\mathcal{P}}$. By the induction on d there is a weak $(1/10r^2)$-net $\tilde{P}'$ for $\tilde{P}$ of size at most $f(10r^2, d - 1)$. The set $P_1' \cup P_2' \cup \tilde{P}'$ is thus a weak $(1/r)$-net whenever

$$\frac{1}{10r^2} \geq \frac{((1/3r)n - 2)^2}{|\tilde{P}|} \geq \frac{(1/9r^2)n^2 - O(n/r)}{n^2/4}.$$

Thus $P_1' \cup P_2' \cup \tilde{P}'$ may fail to be a weak $(1/r)$-net for P only if $n = O(r)$, in which case P itself is a weak $(1/r)$-net of size depending only on r. □

*These notes are from `http://www.borisbukh.org/GeoCombEaster10/suppnotes2.pdf`.

A family $\mathcal{F}$ of convex sets is called a (p,q)-family if among every q members of $\mathcal{F}$ there are p that intersect. Helly theorem asserts that if $\mathcal{F}$ is a $(d+1,d+1)$-family, then there is a point x common to all the member of $\mathcal{F}$. The following result is a version of this for (p,q)-families. A family of hyperplanes in general position shows that (d,d)-family need not to have a point common to many members of $\mathcal{F}$. Also, it might happen that a (p,q_1+q_2-1)-family is a union of a (p,q_1)-family and a (p,q_2)-family that are disjoint from one another, it is no longer possible to guarantee a single point common to all member of a (p,q)-family.

Theorem 3 ((p,q)-theorem)**.** *Let $p \geq d+1$. If $\mathcal{F}$ is a (p,q)-family in $\mathbb{R}^d$, then there is a set $X \subset \mathbb{R}^d$ of size $|X| \leq f(p,q,d)$ such that every $C \in \mathcal{F}$ meets X.*

First we need to establish a weaker result.

Lemma 4. *If $\mathcal{F}$ is a $(d+1,q)$-family in $\mathbb{R}^d$, then there is a point $y \in \mathbb{R}^d$ common to $\varepsilon|\mathcal{F}|$ members of $\mathcal{F}$, where $\varepsilon = \varepsilon(q,d) > 0$.*

Proof. By double-counting the number of $(d+1)$-tuples of sets in $\mathcal{F}$ that have non-empty intersection is at least

$$\frac{\binom{|\mathcal{F}|}{q}}{\binom{|\mathcal{F}|-(d+1)}{q-(d+1)}} \geq \frac{|\mathcal{F}|^q/q^q}{|\mathcal{F}|^{q-(d+1)}} = |\mathcal{F}|^{d+1}/q^q.$$

The existence of the desired point y follows from the fractional Helly theorem. $\square$

Proof of the (p,q)-theorem. As every (p,q)-family is a $(p-1,q)$-family, we can restrict to the case $p=d+1$. By weak ε-net theorem it suffices to find a set $Y \subset \mathbb{R}^d$ that meets every $C \in \mathcal{F}$ in at least $\delta|Y|$ points where $\delta = \delta(q,d)$. Let ε be as in the lemma above. With the hindsight choose δ as to satisfy $2^{1-\delta} > 2-\varepsilon$. For purposes of this proof, let "cloning" of a set mean replacing a set by two copies of itself. Consider the following algorithm to construct Y.

1) Initialise: $Y_0 = \emptyset$, $\mathcal{F}_0 = \mathcal{F}$.

2) At stage $i-1$, pick a point y in at least $\varepsilon|\mathcal{F}|$ members of $\mathcal{F}$.

3) Let $Y_i = Y_{i-1} \cup \{y\}$ and let $\mathcal{F}_i$ be obtained from $\mathcal{F}_{i-1}$ by cloning all the members not containing y.

Repeat for s stages

4) Terminate: output $Y = Y_s$.

Note that $|\mathcal{F}_i| \leq |\mathcal{F}_{i-1}| + (1-\varepsilon)\mathcal{F}_{i-1} = (2-\varepsilon)|\mathcal{F}|$. Thus $|\mathcal{F}_s| \leq (2-\varepsilon)^s|\mathcal{F}|$. If a set $C \in \mathcal{F}$ contains fewer than $\delta|Y|$ members of Y, then it is cloned into at least $2^{|Y|-\delta|Y|} = 2^{(1-\delta)s}$ sets. Thus,

$$2^{(1-\delta)s} \leq (2-\varepsilon)^s|\mathcal{F}|.$$

By the choice of δ it follows that if s is large enough, then there is no set $C \in \mathcal{F}$ containing fewer than $\delta|Y|$ points of Y. $\square$