Geometric combinatorics: supplementary notes 1*

Theorem 1 (Jung's theorem)**.** *If $X \subset \mathbb{R}^d$ is a finite set such that $\|x - y\| \leq 1$ for all $x, y \in X$, then there is a point $p \in \mathbb{R}^d$ such that*

$$\|p - x\| \leq \frac{1}{\sqrt{2}} \left(\tfrac{d}{d+1}\right)^{1/2}$$

for all $x \in P$.

Proof. Since the goal is to prove that the closed balls of radius $\frac{1}{\sqrt{2}} \left(\frac{d}{d+1}\right)^{1/2}$ centred at the points of X have a non-empty intersection, by Helly's theorem it suffices to treat the case $|X| \leq d + 1$.

Without loss of generality, we can assume that the centre of the smallest ball containing X is the origin, and the radius of this ball is r. Let $x_1, \ldots, x_m$ be all the points of X that are at distance exactly r from 0.

The minimality of r implies that $0 \in \operatorname{conv}\{x_1, \ldots, x_m\}$. Indeed, if $0 \notin \operatorname{conv}\{x_1, \ldots, x_m\}$, then there is a vector $v \in \mathbb{R}^d$ such that $\langle v, x_i \rangle > 0$ for all $i = 1, \ldots, m$, and the translated set $P - \varepsilon v$ is closer to the origin than P was, provided ε is small enough. Since $0 \in \operatorname{conv}\{x_1, \ldots, x_m\}$, there are $\alpha_1, \ldots, \alpha_k$ such that

$$\begin{aligned} \alpha_1 x_1 + \cdots + \alpha_m x_m &= 0, \\ \alpha_1 + \cdots + \alpha_k &= 1. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{i \neq j} \alpha_i \geq \sum \alpha_i \|x_i - x_j\|^2 &= \sum_i \alpha_i (2r^2 - 2\langle x_i, x_j \rangle) \\ &= 2r^2 - \langle \sum_i \alpha_i x_i, x_j \rangle = 2r^2. \end{aligned}$$

Averaging over all $j = 1, \ldots, m$ we obtain $\frac{m-1}{m} \sum_i \alpha_i \geq 2r^2$. The result follows from $m \leq d + 1$. □

It is easy to infer from the proof above that the theorem is sharp only if X contains all the $d + 1$ vertices of a regular simplex with side-length 1.

*These notes are from `http://www.borisbukh.org/GeoCombEaster10/suppnotes1.pdf`.

Lemma 2. *If $C \subset \mathbb{R}^d$ is a non-empty compact set, there is a lexicographically minimal point of C.*

Proof. The proof is by induction on d. The base case $d = 0$ is trivial. If $d \geq 1$, let $x_{\min} = \inf\{x_1 : (x_1, \ldots, x_d) \in C\}$. Since $(x_1, \ldots, x_d) \mapsto x_1$ is continuous, the value $x_{\min}$ is attained, and $C' = \{y \in \mathbb{R}^{d-1} : \{x_{\min}\} \times y\}$ is non-empty. Since C' is compact subset of $\mathbb{R}^{d-1}$, the lemma follows by induction. $\square$