Theorem 1 (Jung’s theorem). If \( X \subset \mathbb{R}^d \) is a finite set such that \( \|x - y\| \leq 1 \) for all \( x, y \in X \), then there is a point \( p \in \mathbb{R}^d \) such that
\[
\|p - x\| \leq \frac{1}{\sqrt{2}} \left( \frac{d}{d+1} \right)^{1/2}
\]
for all \( x \in P \).

Proof. Since the goal is to prove that the closed balls of radius \( \frac{1}{\sqrt{2}} \left( \frac{d}{d+1} \right)^{1/2} \) centred at the points of \( X \) have a non-empty intersection, by Helly’s theorem it suffices to treat the case \( |X| \leq d + 1 \).

Without loss of generality, we can assume that the centre of the smallest ball containing \( X \) is the origin, and the radius of this ball is \( r \). Let \( x_1, \ldots, x_m \) be all the points of \( X \) that are at distance exactly \( r \) from 0.

The minimality of \( r \) implies that 0 \( \in \text{conv}\{x_1, \ldots, x_m\} \). Indeed, if 0 \( \not\in \text{conv}\{x_1, \ldots, x_m\} \), then there is a vector \( v \in \mathbb{R}^d \) such that \( \langle v, x_i \rangle > 0 \) for all \( i = 1, \ldots, m \), and the translated set \( P - \varepsilon v \) is closer to the origin than \( P \) was, provided \( \varepsilon \) is small enough. Since 0 \( \in \text{conv}\{x_1, \ldots, x_m\} \), there are \( \alpha_1, \ldots, \alpha_k \) such that
\[
\alpha_1 x_1 + \cdots + \alpha_m x_m = 0,
\]
\[
\alpha_1 + \cdots + \alpha_k = 1.
\]

Hence
\[
\sum_{i \neq j} \alpha_i \geq \sum_{i} \alpha_i \|x_i - x_j\|^2 = \sum_{i} \alpha_i (2r^2 - 2\langle x_i, x_j \rangle)
\]
\[
= 2r^2 - \sum_{i} \alpha_i x_i, x_j = 2r^2.
\]

Averaging over all \( j = 1, \ldots, m \) we obtain \( \frac{m-1}{m} \sum_{i} \alpha_i \geq 2r^2 \). The result follows from \( m \leq d + 1 \).

It is easy to infer from the proof above that the theorem is sharp only if \( X \) contains all the \( d + 1 \) vertices of a regular simplex with side-length 1.

*These notes are from [http://www.borisbukh.org/GeoCombEaster10/suppnotes1.pdf](http://www.borisbukh.org/GeoCombEaster10/suppnotes1.pdf)
Lemma 2. If $C \subset \mathbb{R}^d$ is a non-empty compact set, there is a lexicographically minimal point of $C$.

Proof. The proof is by induction on $d$. The base case $d = 0$ is trivial. If $d \geq 1$, let $x_{\text{min}} = \inf \{x_1 : (x_1, \ldots, x_d) \in C\}$. Since $(x_1, \ldots, x_d) \mapsto x_1$ is continuous, the value $x_{\text{min}}$ is attained, and $C' = \{y \in \mathbb{R}^{d-1} : \{x_{\text{min}}\} \times y\}$ is non-empty. Since $C'$ is compact subset of $\mathbb{R}^{d-1}$, the lemma follows by induction. $\square$